Asymptotic Notation

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1 Preliminaries

Notation: $\exp(x) = e^x$.

Throughout this course we shall use the following shorthand in quantifier notation.

$(\forall a)$ is read as “for all $a$”

$(\exists a)$ is read as “there exists a such that”

$(\forall a, \text{statement}(a))$ is read as “for all a such that statement(a) . . . ”

Example. $(\forall x \neq 0)(\exists y)(xy = 1)$ is a statement which holds in every field but does not hold e.g. in $\mathbb{Z}$, the set of integers, or in $\mathbb{N}$, the set of nonnegative integers.

The symbol $[n]$ will be used to denote $\{1, 2, \ldots, n\}$ in combinatorial contexts.

Definition 1.1 Let $a_n$ be a sequence of real or complex numbers. We write $\lim_{n \to \infty} a_n = c$ (or simply $a_n \to c$) if

$$(\forall \epsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n > n_0)(|a_n - c| < \epsilon).$$

2 Asymptotic Equality

Often, we are interested in comparing the rate of growth of two functions, as inputs increase in length. Asymptotic equality is one formalization of the idea of two functions having the “same rate of growth.”

Definition 2.1 We say $a_n$ is asymptotically equal to $b_n$ (denoted $a_n \sim b_n$) if $\lim_{n \to \infty} a_n / b_n = 1$.

Observations. 1. If $c \neq 0$ is a constant then the statement $a_n \sim c$ is equivalent to $a_n \to c$.

2. If zero occurs infinitely many times in a sequence then that sequence is not asymptotically equal to any sequence, not even to itself.

Exercise 2.2 Let $S$ denote the set of those sequences of real or complex numbers in which zero occurs only a finite number of times. Prove that $\sim$ is an equivalence relation on $S$, i.e., the relation “$\sim$” is
(a) reflexive: \( a \sim a \);
(b) symmetric: if \( a \sim b \) then \( b \sim a \); and
(c) transitive: if \( a \sim b \) and \( b \sim c \) then \( a \sim c \).

**Exercise 2.3** Prove: if \( a_n \sim b_n \) and \( c_n \sim d_n \) then \( a_n c_n \sim b_n d_n \). (Note that a finite number of undefined terms do not invalidate a limit relation.)

**Exercise 2.4** Consider the following statement.

If \( a_n \sim b_n \) and \( c_n \sim d_n \) then \( a_n + c_n \sim b_n + d_n \). (1)

1. Prove that (1) is false.
2. Prove: if \( a_n, b_n, c_n, d_n > 0 \) then (1) is true. *Hint.* Prove: if \( a, b, c, d > 0 \) and \( a/b < c/d \) then \( a/b \prec (a + c)/(b + d) < c/d \).

**Exercise 2.5**

1. If \( f(x) \) and \( g(x) \) are polynomials with respective leading terms \( ax^n \) and \( bx^m \) then \( f(n)/g(n) \sim (a/b)x^{n-m} \).
2. \( \sin(1/n) \sim \ln(1 + 1/n) \sim 1/n \).
3. \( \sqrt{n^2 + 1} - n \sim 1/2n \).
4. If \( f \) is a function, differentiable at zero, \( f(0) = 0 \), and \( f'(0) \neq 0 \), then \( f(1/n) \sim f'(0)/n \). See that items 2 and 3 in this exercise follow from this.

Next we state some of the most important asymptotic formulas in mathematics.

**Theorem 2.6 (Stirling’s Formula)**

\[
\frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \sim n!
\]

**Exercise 2.7** Prove: \( \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}} \).

**Exercise 2.8** Give a very simple proof, without using Stirling’s formula, that \( \ln(n!) \sim n \ln n \).

**Theorem 2.9 (The Prime Number Theorem)** Let \( \pi(x) \) be the number of primes less than or equal to \( x \).

\[
\pi(x) \sim \frac{x}{\ln x},
\]

where \( \ln \) denotes the natural logarithm function.
Exercise 2.10 Let \( p_n \) be the \( n \)-th prime number. Prove, using the Prime Number Theorem, that \( p_n \sim n \ln n \).

Exercise 2.11 Feasibility of generating random prime numbers. Estimate, how many random \( \leq 100 \)-digit integers should we expect to pick before we encounter a prime number? (We generate our numbers by choosing the 100 digits independently at random (initial zeros are permitted), so each of the \( 10^{100} \) numbers has the same probability to be chosen.) Interpret this question as asking the reciprocal of the probability that a randomly chosen integer is prime.

Definition 2.12 A partition of a positive integer \( n \) is a representation of \( n \) as a sum of positive integers: \( n = x_1 + \ldots + x_k \) where \( x_1 \leq \ldots \leq x_k \). Let \( p(n) \) denote the number of partitions of \( n \).

Examples: \( p(1) = 1 \), \( p(2) = 2 \), \( p(3) = 3 \), \( p(4) = 5 \). The 5 representations of 4 are \( 4 = 4; \ 4 = 1 + 3; \ 4 = 2 + 2; \ 4 = 1 + 1 + 2; \ 4 = 1 + 1 + 1 + 1 \). One of the most amazing asymptotic formulas in discrete mathematics gives the growth of \( p(n) \).

Theorem 2.13 (Hardy-Ramanujan Formula)

\[
p(n) \sim \frac{1}{4n\sqrt{3}} \exp \left( \frac{2\pi}{\sqrt{6}} \sqrt{n} \right)
\]

3 Little-oh notation

Definition 3.1 We say that \( a_n = o(b_n) \) (“\( a_n \) is little oh of \( b_n \)”)

if

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 0
\]

Observation. So \( a_n = o(1) \) means \( \lim_{n \to \infty} a_n = 0 \).

Exercise 3.2 Show: if \( a_n = o(c_n) \) and \( b_n = o(d_n) \) then \( a_n \pm b_n = o(c_n) \).

Exercise 3.3 Consider the following statement:

If \( a_n = o(b_n) \) and \( c_n = o(d_n) \) then \( a_n + c_n = o(b_n + d_n) \).

1. Show that statement (3) is false.
2. Prove that statement (3) becomes true if we assume \( b_n, d_n > 0 \).

Exercise 3.4 Show that \( a_n \sim b_n \iff a_n = b_n(1 + o(1)) \).

Exercise 3.5 Use the preceding exercise to give a second proof of (1) when \( a_n, b_n, c_n, d_n > 0 \).
Exercise 3.6 Construct sequences $a_n, b_n > 1$ such that $a_n = o(b_n)$ and $\ln a_n \sim \ln b_n$.

Exercise 3.7 Let $a_n, b_n > 1$. (a) Prove that the relation $a_n = o(b_n)$ does NOT follow from the relation $\ln a_n = o(\ln b_n)$. (b) If we additionally assume that $b_n \to \infty$ then $a_n = o(b_n)$ DOES follow from $\ln a_n = o(\ln b_n)$.

4 Big-Oh, Omega, Theta notation ($O$, $\Omega$, $\Theta$)

Definition 4.1 We say that

1. $a_n = O(b_n)$ ($a_n$ is “big oh” of $b_n$) if $|a_n/b_n|$ is bounded (0/0 counts as “bounded”), i.e.,
   \[(\exists C > 0, n_0 \in \mathbb{N})(\forall n > n_0)(|a_n| \leq C|b_n|).\]

2. $a_n = \Omega(b_n)$ if $b_n = O(a_n)$, i.e., if $|b_n/a_n|$ is bounded ($\exists C > 0, n_0 \in \mathbb{N})(\forall n > n_0)(|a_n| \geq C|b_n|)$

3. $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$, i.e.,
   \[(\exists C, c > 0, n_0 \in \mathbb{N})(\forall n > n_0)(c|b_n| \leq |a_n| \leq C|b_n|).\]

Exercise 4.2 Suppose the finite or infinite limit $\lim_{n \to \infty} |a_n/b_n| = L$ exists. Then

(a) $b_n = o(a_n)$ if and only if $L = \infty$;
(b) $a_n = o(b_n)$ if and only if $L = 0$; and
(c) $a_n = \Theta(b_n)$ if and only if $0 < L < \infty$.

Exercise 4.3 Construct sequences $a_n, b_n > 0$ such that $a_n = \Theta(b_n)$ but the limit $\lim_{n \to \infty} a_n/b_n$ does not exist.

Exercise 4.4 Let $a_n, b_n > 0$. Show: $a_n = \Theta(b_n) \iff \ln a_n = \ln b_n + O(1)$.

Exercise 4.5 Show: if $a_n = O(c_n)$ and $b_n = O(c_n)$ then $a_n + b_n = O(c_n)$.

Exercise 4.6 Consider the statement “if $a_n = \Omega(c_n)$ and $b_n = \Omega(c_n)$ then $a_n + b_n = \Omega(c_n)$.” (a) Show that this statement is false. (b) Show that if we additionally assume $a_n, b_n > 0$ then the statement becomes true.

Exercise 4.7 Let $a_n, b_n > 1$. Suppose $a_n = \Theta(b_n)$. Does it follow that $\ln a_n \sim \ln b_n$?

1. Show that even $\ln a_n = O(\ln b_n)$ does not follow.
2. Show that if $b_n \to \infty$ then $\ln a_n \sim \ln b_n$ follows.

Exercise 4.8 Let $a_n, b_n > 0$. Consider the relations
(A) $a_n = O(2^n)$ and (B) $a_n = 2^O(b_n)$.
(a) Prove: the relation (B) does NOT follow from (A).
(b) Prove: if $b_n > 0.01$ then (B) DOES follow from (A).

Note. $a_n = 2^O(b_n)$ means that $a_n = 2^c n$ where $c = O(b_n)$.

Exercise 4.9 Prove: if $a_n = \Omega(b_n)$ and $a_n = \Omega(c_n)$ then $a_n = \Omega(b_n + c_n)$.

Exercise 4.10 (a) Prove that the relations $a_n = O(b_n)$ and $a_n = O(c_n)$ do NOT imply $a_n = O(b_n + c_n)$.
(b) Prove that if $a_n, b_n > 0$ then the relations $a_n = O(b_n)$ and $a_n = O(c_n)$ DO imply $a_n = O(b_n + c_n)$.

Exercise 4.11 Prove: $\sum_{i=1}^{n} 1/i = \ln n + O(1)$.

5 Prime Numbers

Exercise 5.1 Let $P(x)$ denote the product of all prime numbers $\le x$. Consider the following statement: $\ln P(x) \sim x$. Prove that this statement is equivalent to the Prime Number Theorem.

Exercise 5.2 Prove, without using the Prime Number Theorem, that
$\ln P(x) = \Theta(x)$.

Hint. For the easy upper bound, observe that the binomial coefficient $\binom{2n}{n}$ is divisible by the integer $P(2n)/P(n)$. This observation yields $P(x) \le 4^x$. For the lower bound, prove that if a prime power $p^t$ divides the binomial coefficient $\binom{n}{k}$ then $p^t \le n$. From this it follows that $\binom{2n}{n}$ divides the product $P(2n)P((2n)^{1/2})P((2n)^{1/3})P((2n)^{1/4})\ldots$. Use the upper bound to estimate all but the first term in this product.

6 Partitions

Exercise 6.1 Let $p(n, k)$ denote the number of those partitions of $n$ which have at most $k$ terms. Let $q(n, k)$ denote the number of those partitions in which every term is $\le k$. Observe that $p(n, 1) = q(n, 1) = 1$ and $p(n, n) = q(n, n) = p(n)$.

(Do!) Let $\tilde{p}(n) = \sum_{i=0}^{n} p(i)$ and let $\tilde{p}(n, k) = \sum_{i=0}^{n} p(i, k)$.

1. Prove: $p(n, k) = q(n, k)$.
2. Compute $p(n, 2)$. Give a very simple formula.
3. Compute $p(n, 3)$. Give a simple formula.

4. Prove: $\tilde{p}(n) \leq \tilde{p}(n, k)^2$, where $k = \lfloor \sqrt{n} \rfloor$. Hint. Use part 1 of this exercise.

**Exercise**+ 6.2 Prove, without using the Hardy–Ramanujan formula, that

$$\ln p(n) = \Theta(\sqrt{n}).$$

*Hint.* $\ln p(n) = \Omega(\sqrt{n})$ is easy (2 lines). The upper bound is harder. Use the preceding exercise, especially item 4. When estimating $p(n, \sqrt{n})$, split the terms of your partition into sets $\{x_i \leq \sqrt{n}\}$, $\{\sqrt{n} < x_i \leq 2\sqrt{n}\}$, $\{2\sqrt{n} < x_i \leq 4\sqrt{n}\}$, $\{4\sqrt{n} < x_i \leq 8\sqrt{n}\}$, etc.

**Exercise**+ 6.3 Let $p'(n)$ denote the number of partitions of $n$ such that all terms are primes or 1. Example: $16 = 1 + 1 + 1 + 3 + 3 + 7$. Prove:

$$\ln p'(n) = \Theta\left(\sqrt{\frac{n}{\ln n}}\right).$$

**Exercise** 6.4 Let $r(n)$ denote the number of different integers of the form $\prod x_i!$ where $x_i \geq 1$ and $\sum x_i = n$. (The $x_i$ are integers.) Prove:

$$p'(n) \leq r(n) \leq p(n).$$

**OPEN QUESTIONS.** Is $\log r(n) = \Theta(\sqrt{n})$? Or perhaps, $\log r(n) = \Theta(\sqrt{n}/\log n)$? Or maybe $\log r(n)$ lies somewhere between these bounds?