1. (8 points) Let $G$ be a connected planar graph such that every vertex has degree $\geq 3$. John claims to have drawn $G$ in the plane such that each region is 6-sided. Prove him wrong.

2. (6 points) Prove that every DAG can be topologically sorted. In other words: if $G$ is an acyclic digraph, prove that the vertices of $G$ can be numbered such that if $i \rightarrow j$ is an edge then $i < j$.

3. (9 points) Prove that almost all tournaments are strongly connected. In other words, let $p_n$ denote the probability that a random tournament on a given set of $n$ vertices is strongly connected. Prove that $p_n \rightarrow 1$. State the size of the sample space for this experiment.

4. (7+3 points) Let $A$ be the adjacency matrix of a DAG $G$ with $n$ vertices. (a) Determine the rank of $A^n$. Prove your answer. (b) Show that the rank of $A^{n-1}$ determines whether or not $G$ has a Hamilton path.

5. (9+3 points) Let $G$ be a random graph on the vertex set $\{1, 2, \ldots, n\}$ (each pair of vertices is adjacent with probability $1/2$). Let $p_n$ denote the probability of the event that $\deg(1) = \deg(2)$. (a) Give a closed-form expression for $p_n$. (b) Prove that $p_n \sim c/\sqrt{n}$. Determine the value of the constant $c$.

6. (1+5+2+8+3 points) Let $T_n$ denote the number of triangles in a random graph. (a) State the size of the sample space in this experiment. (b) Write $T_n$ as a sum of indicator variables. Give a clear definition of each indicator variable in the sum and state the number of indicator variables used. (c) Determine $E(T_n)$. (d) Determine $\text{Var}(T_n)$. Give a closed-form expression. (e) Asymptotically evaluate $\text{Var}(T_n)$: show that $\text{Var}(T_n) \sim cn^d$; determine the constants $c$ and $d$.

7. (Bonus problem, not required, 6 points) Let $G$ be a directed graph such that all vertices have out-degree $\leq k$. Prove that $G$ is $(2k+1)$-colorable. (A legal coloring of a directed graph is defined the same way as for undirected graphs; coloring is insensitive to the direction of edges.)

8. (Bonus problem, not required, 6 points) Prove that for almost all graphs, every vertex has degree $n/2 + O(\sqrt{n \log n})$. 

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*A tournament* is an orientation of the complete graph, i.e., every edge of the complete graph is oriented in one of the two possible directions. In a *random tournament*, these orientations are assigned independently with each choice having probability $1/2$. 