1. (4 points) Prove: if $0 \leq k < n/2$ then $\binom{n}{k} \leq \binom{n}{k+1}$.

2. (6 points) Suppose you receive equal amounts of spam email and non-spam email. Further suppose the probability that a spam email contains the word “free” is $\frac{1}{3}$ and the probability that a non-spam e-mail contains the word “free” is $\frac{1}{30}$. Your software tells you that you received an e-mail that contains the word “free.” What is the probability that the email is spam?

3. (18 points) Let $p$ be a prime number and let $f(x) = 1+x+x^2+\cdots+x^{p-2}$. Prove: $(\forall x)(f(x) \equiv -1, 0 \text{ or } 1 \mod p)$.

4. (6+14 points) (a) Count the increasing functions $f: [k] \to [n]$. (Recall the notation $[k] = \{1, \ldots, k\}$.) (b) Count those functions $f: [k] \to [n]$ that satisfy $f(i+1) \geq f(i) + 2$ for every $i$. Your answers should be simple expressions involving binomial coefficients. Prove your answers.

5. (5+5+5 points) (a) Find a sequence $a_n$ such that $\lim_{n \to \infty} a_n = 1$ but $\lim_{n \to \infty} a_n^n = \infty$. (b) Find a sequence $b_n$ such that $\lim_{n \to \infty} b_n = \infty$ but $b_{n+1} \sim b_n$. (c) Prove: if $c_n > 1$ and $c_{n+1} = O(c_n)$ then $\ln c_n = O(n)$.

6. (3+5+2 points) Let $I$ denote the $n \times n$ identity matrix and $J$ the $n \times n$ all-ones matrix (every entry is 1). Let $A = I + J$.

   (a) For the case $n = 3$ write down in full detail the system $Ax = 0$ of homogeneous linear equations. Call the unknowns $x_1, x_2, x_3$. Using this notation, what does $x$ mean?

   (b) For every $n$, prove: the columns of $A$ are linearly independent.

   (c) What does item (b) say about the solutions of the system $Ax = 0$?

7. (5+3+5+3B+5B+4B points) Let $n \geq 3$ and let $A$ be an $n \times n$ matrix with characteristic polynomial $f_A(t) = t^n - 3t + 2$.

   (a) Decide whether or not $A$ is singular. Clearly say YES or NO. Prove your answer. (Recall that an $n \times n$ matrix is singular if $\det(A) = 0$.)

   (b) Prove that $A$ has an eigenvalue that is an integer.
(c) Find the sum of all the $n$ (complex) eigenvalues of $A$. Indicate the facts you use to obtain your answer.

(d) (BONUS) Prove: $A$ is not a stochastic matrix.

(e) (BONUS) Prove: If $n \geq 4$ then $A$ is diagonalizable.

(f) (BONUS) Prove: Disprove the conclusion of (c) when $n = 3$.

8. (6+4B+4B points) Let $V$ be an $n$-dimensional euclidean space.

(a) Let $v_1, \ldots, v_k$ be pairwise orthogonal non-zero vectors in $V$. Prove that they are linearly independent.

(b) (BONUS) Prove that the functions $\cos(t), \cos(2t), \ldots, \cos(kt)$ are linearly independent.

(b) (BONUS) Let $U$ be a $k$-dimensional subspace of $V$. Let $U^{\perp}$ denote the set of those vectors that are orthogonal to all vectors in $U$. Note that $U^{\perp}$ is a subspace. (You don’t need to prove this.) Prove: $\dim(U^{\perp}) = n - k$.

9. (6+4+5+5+4B points) Let $A$ be a real symmetric $n \times n$ matrix. We say that $A$ is positive semidefinite if $(\forall x \in \mathbb{R}^n)(x^\dagger Ax \geq 0)$ (where $\dagger$ indicates transpose). We say that $A$ is positive definite if $(\forall x \in \mathbb{R}^n)(x \neq 0 \Rightarrow x^\dagger Ax > 0)$

(a) Prove: Prove that the all-ones matrix $J$ is positive semidefinite.

(b) Prove: $I + J$ is positive definite.

(c) Prove: If $A$ is positive definite then $A$ is non-singular.

(d) Prove: if $A$ is positive semidefinite then all eigenvalues of $A$ are non-negative.

(e) (BONUS) State and prove the converse of (d).

10. (BONUS: 5B points) Let $A \in M_n(\mathbb{R})$ be a symmetric real matrix. Prove that all of its eigenvalues (over $\mathbb{C}$) are real. Do not use the Spectral Theorem.

11. (6+8+5+6+6B points)

(a) Draw the diagram of a weakly connected finite Markov Chain which has more than one stationary distribution. Give two stationary distributions for your Markov Chain. Use as few states as possible.

(b) Recall that a finite Markov Chain is irreducible if its transition digraph is strongly connected. Draw the diagram of a reducible (not irreducible) finite Markov Chain with a unique stationary distribution. State the stationary distribution. Do not prove. Use as few states as possible.
(c) Define ergodicity of a finite Markov Chain. Define the terms used in the definition in terms of directed graph concepts.

(d) Draw the diagram of an irreducible but non-ergodic finite Markov Chain.

(e) (BONUS) Prove: the stationary distribution of an irreducible finite Markov Chain is unique. (Prove uniqueness only. Do not prove the existence of a stationary distribution.) Do not use the Frobenius-Perron Theorem.

12. (15+7 points)

(a) Let $A \in M_n(\mathbb{R})$ be a symmetric real matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\rho = \max\{|\lambda_i| : 1 \leq i \leq n\}$. Prove: $(\forall v \in \mathbb{R}^n)(\|Av\| \leq \rho\|v\|)$. 

*Hint:* Recall that the Spectral Theorem says that $A$ has an orthonormal eigenbasis. Represent $v$ as a linear combination of this basis. Recall that $\|v\|^2 = v^\dagger v$.

(b) Find a real $2 \times 2$ matrix $B$ with real eigenvalues and a vector $v \in \mathbb{R}^2$ such that $\|Bv\| > \rho\|v\|$. State the eigenvalues and the value $\rho$ for your matrix. 

*Hint:* Make $B$ triangular.

13. (5+15+5 points) We roll $n$ dice; the numbers shown are $X_1, \ldots, X_n$. 

$(1 \leq X_i \leq 6.)$ Let $Y = \sum_{i=1}^{n-1} X_i X_{i+1}$. Compute (a) $E(Y)$ (b) $\text{Var}(Y)$.

(c) Asymptotically evaluate $\text{Var}(Y)$. Your answers to each question should be simple closed-form expressions.

14. (3+4+4+5B+5B points) Let $(\Omega, P)$ be a probability space with $|\Omega| = n$.

(a) What do we call the elements of the function space $\mathbb{R}^\Omega$?

(b) What is the dimension of $\mathbb{R}^\Omega$? Describe a basis.

(c) Let $X_1, \ldots, X_k$ be pairwise independent random variables such that $(\forall i)(E(X_i) = 0$ and $\text{Var}(X_i) = 1)$. For all $i$ and $j$, determine $E(X_i X_j)$.

(d) (BONUS) Prove: under the assumptions of (c), the random variables $X_1, \ldots, X_k$ are linearly independent.

(e) (BONUS) Prove: If there exist $k$ non-trivial, pairwise independent events in $(\Omega, P)$ then $n \geq k + 1$.

15. (5+6+10 points) (a) Draw a topological $K_{3,3}$ with 10 vertices. (b) State Kuratowski’s characterization of planar graphs. (c) Prove: if a connected graph has $n$ vertices and $n + 2$ edges then it is planar.

16. (18 points) Prove: for all sufficiently large $n$, the probability that a random graph is planar is less than $2^{-0.49n^2}$. 

3
17. **(20 points)** Prove: almost all graphs on \( n \) vertices have no clique (complete subgraph) of size \( \geq 1 + 2 \log_2 n \). (Hint: estimate the probability of cliques of size \( k \). Do not substitute the value \( 1 + 2 \log_2 n \) for \( k \) until the very end to avoid messy formulas.)

18. **(8+12 points)** (a) State the multinomial theorem: express \((x_1 + \cdots + x_k)^n\) as a sum. Express the coefficients in terms of factorials.
   (b) Count the terms in your expression. Your answer should be a very simple expression (a binomial coefficient).

19. **(1+7 points)** (a) Define the little-o notation.
   (b) Prove: \( n^{100} = o(1.01^n) \). Elegance counts. Do not use L'Hospital's rule beyond using the fact that \( \lim_{x \to \infty} \frac{\ln x}{x} = 0 \). (Hint: substitute a new variable.)

20. **(1+9 points)** (a) Count the strings of length \( n \) over the alphabet \{A, B, C, D, E\}. (b) How many among these strings use all the five letters? Your answer should be a closed-form expression.

21. **(8 points)** Construct a probability space and two random variables that are uncorrelated but not independent. Make your sample space as small as possible. (Recall: the random variables \( X, Y \) are uncorrelated if \( E(XY) = E(X)E(Y) \).)

22. **(5+15 points)** (a) Prove: if \( p \) is a prime then the only solutions to the congruence \( x^2 \equiv 1 \pmod{p} \) are \( x \equiv \pm 1 \pmod{p} \). (b) Let \( p < q < r \) be three distinct odd primes. Let \( n = pqr \). Count the solutions to the congruence \( x^2 \equiv 1 \pmod{n} \). (Two solutions count as distinct if they are not congruent modulo \( n \).) Prove your answer.

23. **(10+10 points)** Let \( a_n > 2 \) and \( b_n > 2 \) be sequences of real numbers. Consider the following two statements: (1) \( a_n = \Theta(b_n) \); (2) \( \ln a_n \sim \ln b_n \). (a) Prove that (2) does not follow from (1). (b) Prove that if \( a_n \to \infty \) then (2) follows from (1).

24. **(2+8 points)** Let \( X \) be a random nonnegative integer with 100 decimal digits; initial zeros are permitted. (Each of the 100 digits is chosen at random from \{0, 1, \ldots, 9\}. (a) What is the size of the sample space of this experiment? (b) Estimate the probability that \( X \) is prime. Use the approximation \( \ln 10 \approx 2.303 \). Do not use a calculator. Your answer should be a simple fraction.

25. **(BONUS 6B points)** Let \( n = pq \) where \( p, q \) are distinct primes. Prove that the following statement is false:
   \((\forall a)(\text{if } \gcd(a, n) = 1 \text{ then } a^{n-1} \equiv 1 \pmod{n})\).