1. (5 points) Draw a strongly connected digraph with period 3 that has no directed cycle of length 3. Use as few edges as possible; state the number of edges you use.

2. (6 points) Prove: if a finite Markov Chain has two stationary distributions then it has infinitely many.

3. (15 points: 1 point for each answer, 3+4+5 points for the proofs)
   True or false (circle one, prove). (a) If two vectors are linearly dependent then one of them is a scalar times the other. T F
   (b) If the determinant of a $3 \times 3$ matrix is zero then one of the columns is a scalar times another column. T F
   (c) If $A$ and $B$ are $3 \times 3$ matrices with positive entries then
   \[
   \text{rk}(A + B) \geq \text{rk}(A). \quad T \quad F
   \]
4. (7 points) Find the characteristic polynomial and the eigenvalues of the
matrix \( B = \begin{pmatrix} 5 & 2 \\ 4 & 7 \end{pmatrix} \).

5. (7 points) Find an \( n \times n \) matrix \( A \) such that \( \text{rk}(A) = 1 \) and all the \( n^2 \)
entries of \( A \) are distinct.

6. (10 points) Let \( \alpha_1, \ldots, \alpha_n \) be distinct real numbers. Let
\( f(x) = \prod_{i=1}^{n} (x - \alpha_i) \) and let \( g_i(x) = f(x)/(x - \alpha_i) \). (So each \( g_i \) is a poly-
nomial of degree \( n - 1 \).) Prove that \( g_1, \ldots, g_n \) are linearly independent
in the space \( \mathbb{R}[x] \) of polynomials over \( \mathbb{R} \).

7. (BONUS: 4B points) Let \( A \) be an \( n \times n \) matrix with integer entries.
Suppose all diagonal entries are odd and all other entries are even.
Prove that \( A \) is non-singular (i.e., \( \det(A) \neq 0 \)).

8. (BONUS: 4B points) Prove: every non-singular \( n \times n \) matrix can be
turned into a singular matrix by changing just one entry.

9. (BONUS: 2B points) Let \( A, B \) be \( n \times n \) stochastic matrices. Prove:
\( A - B \) is singular.