1. (20+20+10B points) Let $A$ be an $n \times n$ matrix and $\lambda$ an eigenvalue. Recall: the geometric multiplicity of $\lambda$ is the maximum number of linearly independent eigenvectors to eigenvalue $\lambda$. The algebraic multiplicity of $\lambda$ is $k$ if the characteristic polynomial $f_A(t)$ is divisible by the polynomial $(t - \lambda)^k$ but not divisible by the polynomial $(t - \lambda)^{k+1}$.

(a) Determine (a1) the geometric and (a2) the algebraic multiplicity of the eigenvalue $\lambda = 2$ for the matrix

$$A = \begin{pmatrix} 17 & 13 & 11 & 7 & 5 \\ 0 & 2 & 1 & -7 & 5 \\ 0 & 0 & 2 & 3 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}$$

(Hint: Use the Rank-Nullity Theorem to determine the geometric multiplicity.) **Show your work.** Result without details of the calculation will not be accepted. **DO NOT USE** electronic devices.

(b) (BONUS) Prove: For any $n \times n$ matrix $B$ and any eigenvalue $\lambda$ of $B$, the geometric multiplicity of $\lambda$ is always $\leq$ the algebraic multiplicity.

2. (5+15+20+5 points) For each of the following relations on the universe specified, determine whether or not it is an equivalence relation. Clearly answer YES or NO. Prove your answers.

(a) Universe: the integers $\geq 2$. Relation: “not relatively prime.”

(b) Universe: all integers. Relation: “$x^2 \equiv y^{14} \text{ (mod 7)}$.”

(c) Universe: all non-trivial events in the uniform probability space over a sample space of size $n$. Relation: “not independent.” Your answer should depend on $n$.

(d) Universe: all infinite sequences of real numbers. Relation: “$a_n - b_n = O(n)$.”
3. (8+20+10 points) Let $A$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\mu = \max_i |\lambda_i|$.

(a) The Spectral Theorem says that $A$ has an orthonormal eigenbasis. What does this mean? Define “orthonormal eigenbasis.” State how many vectors are in an orthonormal eigenbasis of $A$.

(b) Prove: For every $v \in \mathbb{R}^n$, we have $\|Av\| \leq \mu \|v\|$. (Use the Spectral Theorem but no other theorems that we did not prove in class. Recall that the norm of $v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$ is defined as $\|v\| = \sqrt{v^Tv} = \sqrt{\sum v_i^2}$. ($T$ stands for transpose.)

(c) Prove: There exists $w \in \mathbb{R}^n$, $w \neq 0$ such that $\|Aw\| = \mu \|w\|$.

4. (16 points) A graph $G$ has $n$ vertices, out of which $k$ vertices have degree 7 and the remaining $(n - k)$ vertices have degree 16. Determine the number of paths of length 2 in $G$. (Note: the path $u - v - w$ and the path $w - v - u$ count as the same path, so for instance $K_3$ has 3 paths of length 2.)

5. (9+9 points) Let $\varphi : V \to W$ be a linear map. For each of the following statements, decide whether or not the statement is true for every $\varphi$.

(a) If the vectors $v_1, \ldots, v_k \in V$ are linearly independent then their images, $\varphi(v_1), \ldots, \varphi(v_k) \in W$ are linearly independent.

(b) If the vectors $v_1, \ldots, v_k \in V$ are linearly dependent then their images, $\varphi(v_1), \ldots, \varphi(v_k) \in W$ are linearly dependent.

If your answer is “YES,” prove. If your answer is “NO,” give a specific counterexample (define $V, W, \varphi$, and the vectors $v_1, \ldots, v_k$).

6. (12+15+15+6B points) Consider the following random walk on the number line. $X_t$ denotes the position of our wandering particle at time $t$. We start at the origin: $X_0 = 0$; and then at each time step, the particle moves one step to the right with probability $2/3$ or one step to the left with probability $1/3$. Formally: $P(X_{t+1} = j+1 \mid X_t = j) = 2/3$ and $P(X_{t+1} = j-1 \mid X_t = j) = 1/3$.

(i) Determine $E(X_t)$.

(ii) Determine $\text{Var}(X_t)$.

(iii) Determine the probability $p_t(j) = P(X_t = j)$. Your answer should be a simple closed-form expression.

(iv) (BONUS) What is the most likely position of the particle at time $t$? Call this position $j_t$, so $p_t(j_t) = \max_j p_t(j)$. Prove: $|j_t - E(X_t)| \leq 1$.

7. (3+15 points) Select a random integer $X$ with $n$ digits such that all digits are odd (i.e., the digits are from the set $\{1, 3, 5, 7, 9\}$).
(a) What is the size of the sample space for this experiment?
(b) What is the probability that each of the 5 odd digits actually occur in $X$? Your answer should be a closed-form expression (no summation symbols or dot-dot-dots).

8. (20 + 5 points) Let $G$ be a graph with $n$ vertices and $\leq n$ edges.

(a) Prove: $\chi(G) = O(\sqrt{n})$. ("$\chi$" denotes the chromatic number.)
(b) Prove that this bound is tight, i.e., construct an infinite family of graphs satisfying the condition and having chromatic number $\chi(G) = \Omega(\sqrt{n})$.

9. (2+12 points) We flip 5 fair coins. Let $X_i$ be the indicator variable of the event that the $i$-th coin comes up “Heads.”

(a) State the size of the sample space for this experiment.
(b) What is the probability of the event $X_1 = X_2X_3 + X_4X_5$? Show all your work.

10. (20 points) Consider a Bernoulli trial with probability $p$ of success, i.e., we flip a biased coin that comes up Heads with probability $p$ and Tails with probability $1-p$; “Heads” counts as “success.” We keep flipping the coin until the first success. Let $X$ denote the number of times we flipped the coin. Determine $E(X)$.

11. (18 points; lose up to 4 points for each mistake) Recall that an $n \times n$ matrix $A$ is non-singular if an only if the columns of $A$ are linearly independent. State five conditions that are equivalent to this: “$A$ is non-singular if and only if …”. Complete the sentence with a statement involving the concept in parentheses in each case.

(a) (determinant)
(b) (rank)
(c) (solutions to system of linear equations [which system?])
(d) (eigenvalues)
(e) (inverse)

12. (12 points) Let $A = (a_{ij})$ be an $n \times n$ matrix with integer entries. Assume each diagonal entry $a_{ii}$ is odd and each off-diagonal entry $a_{ij}$ ($j \neq i$) is even. Prove: $A$ is nonsingular.

13. (12 points) Asymptotically evaluate the binomial coefficient $\binom{3n}{n}$. Your answer should be of the form $\binom{3n}{n} \sim an^b e^n$ where $a, b, c$ are constants. Determine $a, b, c$.

14. (12 points) Let $G = (V, E)$ and $H = (V, F)$ be two graphs with the same vertex set, $V$. Let $L = (V, E \cup F)$. Prove: $\chi(L) \leq \chi(G)\chi(H)$. ("$\chi$" denotes the chromatic number.)
15. (12 points) Prove: if a finite Markov Chain has two stationary distributions then it has infinitely many.

16. (8 points) Evaluate this expression in closed form:
\[
\sum_{k=0}^{n} \binom{n}{k} 2^{-k/2}.
\]

17. (BONUS 8 points) Recall: an \( n \times n \) matrix is stochastic if it is the transition matrix of a finite Markov Chain. Prove: if \( \lambda \) is a (real or complex) eigenvalue of a stochastic matrix then \( |\lambda| \leq 1 \). (If you are uncomfortable with complex numbers, assume \( \lambda \) is real for 8 points.)

18. (BONUS 1+3+5 points) Let \( A = (a_{ij}) \) be the adjacency matrix of the graph \( G = (V,E) \) where \( V = [n] = \{1,\ldots,n\} \), so \( a_{ij} = 1 \) if the vertices \( i \) and \( j \) are adjacent and 0 otherwise. Let \( \lambda_1 \geq \cdots \geq \lambda_n \) be the eigenvalues of \( A \). Express (a) \( \sum_i \lambda_i \)  (b) \( \sum_i \lambda_i^2 \)  (c) \( \sum \lambda_i^3 \) in terms of simple combinatorial parameters of \( G \) such as the number of certain small subgraphs.

19. (BONUS 6B points) Prove:
\[
\sum_{i=0}^{k} \binom{n}{i} \leq \left( \frac{ne}{k} \right)^k.
\]

20. (BONUS 3B points) A graph is self-complementary if it is isomorphic to its complement. Prove: if \( G \) is self-complementary then \( \chi(G) \geq \sqrt{n} \). (\( n \) is the number of vertices.)

21. (BONUS 3B points) Prove that the \( 3 \times 3 \times 3 \) grid graph has no Hamilton path ending in the center. (A Hamilton path is a path that includes every vertex.) (The graph in question is a 3-dimensional grid; it has 27 vertices.)

22. (BONUS 2B points) Let \( A, B \in M_n(\mathbb{R}) \) (real \( n \times n \) matrices). Prove: \( AB - BA \neq I \) (where \( I \) is the identity matrix).