Name (print): ________________________________

Do not use book, notes, scratch paper. Show all your work. If you are not sure of the meaning of a problem, ask the instructor. The bonus problems are underrated, do not work on them until you are done with everything else. Write your solution in the space provided. You may continue on the reverse. This exam contributes 5% to your course grade.

1. (4+7+7+6+4 points) A polynomial of degree at most 3 is an expression of the form $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_i \in \mathbb{R}$. Let $P_3$ denote the set of these polynomials. Note (do not prove) that $P_3$ is a vector space.

(a) State dim($P_3$) and list a basis of $P_3$. Do not prove.

(b) Consider the set $T = \{f \in P_3 \mid f(1) = 0\}$. Note (do not prove) that this is a subspace of $P_3$. State dim($T$) and list a basis of $T$. Do not prove.

(c) Prove that the following four polynomials are linearly dependent:

- $f_1(x) = x^3 + 3x^2 + 3x + 1$,
- $f_2(x) = x^3 + 1$,
- $f_3(x) = x^2 + 3x + 2$,
- $f_4(x) = (x^2 - 1)(x + 17)$. Your proof should not involve any calculation.

(d) Let $U$ be the span of the four polynomials from part (c) and let $g(x) = x^3 + x + 1$. Prove: $g \notin U$. Your proof should not involve any calculation.

(e) Let $D : P_3 \to P_3$ be the differentiation operator: $D(f) = f'$. (For instance, $D(f_3) = 2x + 3$.) Note (do not prove) that this is a linear transformation of $P_3$. Find the eigenvalues and eigenvectors of $D$. Prove that there are no eigenvectors other than what you found.
2. (4 points) Let $a_n$ and $b_n$ be sequences of positive numbers. TRUE or FALSE (circle the right answer): $a_n = o(b_n) \implies a_n = o(b_n^2)$. (Note the little-oh notation.) If “true,” do not prove. If “false,” give a concrete counterexample.

3. (6 points) Define linear independence. Your definition should be the phrase “The vectors $v_1, \ldots, v_k$ are linearly independent if” followed by a properly quantified formula with no English words. State the formula.

4. (6 points) Compute the characteristic polynomial and the eigenvalues of the matrix

\[
\begin{pmatrix}
3 & 4 \\
2 & -1
\end{pmatrix}
\]

5. (6 points) Draw the diagram (digraph with transition probabilities) of a weakly connected Markov Chain that has more than one stationary distribution. State two stationary distributions. Use as few states as possible.

6. (BONUS: 3B points) Let $A \in M_3(\mathbb{R})$ (a $3 \times 3$ matrix). Prove: $A$ has an eigenvector in $\mathbb{R}^3$.

7. (BONUS: 4B points) Find a $3 \times 3$ stochastic matrix that has one real and two non-real complex eigenvalues.

8. (BONUS: 6B points) Let $A_1, \ldots, A_m$ be events such that $(\forall i)(P(A_i) = 1/2)$ and $(\forall i \neq j)(P(A_i \cap A_j) \leq 1/5)$. Prove: $m \leq 6$. (Hint: Let $X_i$ be the indicator variable of $A_i$. Compute $\text{Var}(\sum X_i)$.)