1. (a) (5 points) Let $A$ be a comparison-based sorting algorithm (input: a list of reals, output: sorted list). Prove: $A$ makes $\gtrsim n \log n$ comparisons on the worst input of size $n$ (size = number of items to be sorted).

(b) (G only, 4 points) (Average-case analysis) Suppose $A$ receives “random” inputs (every permutation of a given list of reals is equally likely to be the input). Prove: if $A$ has at least 1% chance of returning a correctly sorted list then $A$ requires $\gtrsim n \log n$ comparisons on the worst input.

2. (8 points) (Binary search) Given an array $A[1 \ldots n]$ of reals, decide whether or not a given real number $x$ is in the array. Do it with asymptotically optimum efficiency. Describe the algorithm in pseudocode. State the number of comparisons made.
3. (All-ones square problem.)

(a) (10 points) Given an \( n \times n \) array \( A \) of zeros and ones, find the maximum size of a contiguous square of all ones. (You do not need to locate such a largest all-ones square, just determine its size.) Solve this problem in linear time. “Linear time” means the number of steps must be \( O(\text{size of the input}) \). In the present problem, the size of the input is \( O(n^2) \). Manipulating integers between 0 and \( n \) counts as one step; such manipulation includes copying, incrementing, addition and subtraction, looking up an entry in an \( n \times n \) array. Describe your solution in pseudocode. The solution should be very simple, no more than a few lines. Elegance counts. Define your auxiliary variables. Remember: the clear definition (the “brain” of the solution) accounts for 50% of the grade.

(b) (G only, 4 points) Modify the pseudocode so it will return the location of a maximum size all-ones square.

4. (a) (2 points) For two sequences of real numbers, \( \{a_n\} \) and \( \{b_n\} \), define the relation \( a_n \sim b_n \) (“\( a_n \) is asymptotically equal to \( b_n \)”) as in the handout.

(b) (4 points) Explain why the statement “\( (\forall n)(a_n \sim b_n) \)” does not make sense. Your explanation should be short and to the point.

(c) (4 points) For two sequences of real numbers, \( \{a_n\} \) and \( \{b_n\} \), define the relation \( a_n \gtrsim b_n \) (“\( a_n \) is greater than or asymptotically equal to \( b_n \)”).

(d) (4 points) Give an example of two sequences of real numbers, \( \{a_n\} \) and \( \{b_n\} \), such that \( a_n \gtrsim b_n \) but infinitely often \( a_n < b_n \) and yet \( a_n \neq b_n \).

(e) (G only, 8 points) Let \( \{a_n\} \) \( \{b_n\} \), and \( c_n \) be sequences of real numbers. Prove: if \( (\forall n)(a_n \geq b_n) \) and \( b_n \sim c_n \) then \( a_n \gtrsim c_n \). (Use the back side of this sheet.)