1 Introduction

Notation. \([n]\) denotes the set \(\{1, 2, \ldots, n\}\)

Exercise 1.1 Let \(4 \mid n\) and let \(A_1, \ldots, A_m \subset [n]\) such that for all \(i \neq j\),
\[|A_i \cap A_j| = n/4.\] Prove: \(m \leq n - 1\). Hint. Use linear algebra.

Exercise 1.2 Prove that the inequality \(m \leq n - 1\) in the preceding exercise is tight, i.e., for infinitely many values of \(n\), set systems as described in the preceding exercise exist with \(m = n - 1\). Hint. Does this problem belong in these notes?

Definition 1.3 A \((\pm 1)\)-matrix is a matrix whose entries are 1 and \(-1\).

An \(n \times n\) \((\pm 1)\)-matrix is called an Hadamard matrix if the rows are orthogonal.

Remark. In Hadamard’s name, the “H” and the final “d” are silent.

Exercise 1.4 Prove that an \(n \times n\) \((\pm 1)\)-matrix \(H\) is Hadamard \(\iff H \cdot H^t = nI_n\), where \(I_n\) denotes the \(n \times n\) identity matrix.

Definition 1.5 An \(n \times n\) real matrix is orthogonal if \(AA^t = I_n\).

Exercise 1.6 A real \(n \times n\) matrix \(A\) is orthogonal \(\iff (\forall x \in \mathbb{R}^n)(\|Ax\| = \|x\|),\)
where \(\|x\| = \sqrt{x \cdot x^t}\) denotes the Euclidean norm.

Exercise 1.7 Prove: if \(H\) is an \(n \times n\) Hadamard matrix then \(\frac{1}{\sqrt{n}} H\) is an orthogonal matrix.

Exercise 1.8 If \(H\) is an \(n \times n\) Hadamard matrix then \((\forall x \in \mathbb{R}^n)(\|Hx\| = \sqrt{n}\|x\|)).\)

Exercise 1.9 Prove that the columns of an Hadamard matrix are also orthogonal, i.e., \(H^t \cdot H = nI_n\).
Exercise 1.10 Prove: all (complex) eigenvalues of an \( n \times n \) Hadamard matrix have absolute value \( \sqrt{n} \).

Exercise 1.11 Prove: if \( H \) is an \( n \times n \) Hadamard matrix then \( \det(H) = \pm n^{n/2} \).

Exercise 1.12 Prove: if \( A \) is an \( n \times n \) \((\pm 1)\)-matrix then \( |\det(A)| \leq n^{n/2} \). Equality holds if and only if \( A \) is an Hadamard matrix. \( \text{Hint.} \) Prove Hadamard’s Inequality: if \( A \) is an \( n \times n \) real matrix then \( |\det(A)| \leq N_1 \cdot \ldots \cdot N_n \) where \( N_i \) is the Euclidean norm of the \( i^{th} \) row of \( A \). Equality holds exactly when either a row is zero or the rows are orthogonal. Use the geometric meaning of the determinant (volume of the paralelloped spanned by the rows).

Example 1.13 \( S_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), \( S_{k+1} = \begin{bmatrix} S_k & S_k \\ S_k & -S_k \end{bmatrix} \) (\( k \geq 1 \)).

The matrix \( S_k \) is called the \( 2^k \times 2^k \) Sylvester matrix.

Exercise 1.14 Prove that \( S_k \) is an Hadamard matrix.

Exercise 1.15 Let \( a_{v,w} = (-1)^{v \cdot w} \), where \( v, w \in \mathbb{F}_2^k \). Prove that the \( 2^k \times 2^k \) matrix \((a_{v,w})\) is \( S_k \) (after suitable renumbering of the rows and columns).

Definition 1.16 The group \( \mathbb{Z}_n^2 = \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \) is called an elementary Abelian 2-group.

Remark. This group is the additive group of the \( n \)-dimensional vector space over \( \mathbb{F}_2 \). \( \mathbb{F}_2 \) is the field of two elements.) \( \mathbb{Z}_n^2 \) is also the additive group of the field \( \mathbb{F}_{2^n} \).

Exercise 1.17 Prove that the Sylvester matrix \( S_k \) is the character table of \( \mathbb{Z}_n^2 \).

Exercise 1.18 Let \( p \) be a prime number, \( p \equiv -1 \pmod{4} \). Construct a \( (p + 1) \times (p + 1) \) Hadamard matrix using the quadratic character of the field \( \mathbb{F}_p \). \( \text{Hint.} \) Consider the \( p \times p \) matrix \((a_{ij})\) where \( a_{ij} = \left( \frac{i + j}{p} \right) \) (Legendre symbol). Modify this matrix by adding a row and column and suitably changing the zeros to \( \pm 1 \).

Exercise 1.19 Prove: if \( \exists n \times n \) Hadamard matrix, then \( n = 2 \) or \( 4 \mid n \).

Exercise 1.20 Prove: if \( \exists k \times k \) Hadamard matrix and \( \exists l \times l \) Hadamard matrix, then \( \exists kl \times kl \) Hadamard matrix. \( \text{Hint.} \) Kronecker product.

Comment. The Sylvester matrices are Kronecker powers of \( S_1 \): \( S_k = S_1 \otimes \cdots \otimes S_1 \).
Conjecture 1.21  If $4 \mid n$, then $\exists$ an $n \times n$ Hadamard matrix.

Comment. Let $\mathcal{H} = \{ n \mid \exists n \times n$ Hadamard matrix $\}$ and let $h_n = |\mathcal{H} \cap [n]|$. If the conjecture is true, then $h_n = \Omega(n)$. But even this weak consequence of the conjecture remains unsolved.

Open Problem 1.22  Prove that $h_n \neq o(n)$.

Exercise 1.23  Prove that $h_n = \Omega\left(\frac{n}{\log(n)}\right)$.

2 Discrepancy and Ramsey Theory for $(\pm 1)$-Matrices

Lemma 2.1 (Lindsey’s Lemma)  Let $H = (h_{ij})$ be a Hadamard matrix. Let $S, T \subseteq [n]$ and $s = |S|$, $t = |T|$. Then

$$\left| \sum_{i \in S} \sum_{j \in T} h_{ij} \right| \leq \sqrt{stn}.$$

Definition 2.2  We call the submatrix on the entries corresponding to $S \times T$ an $s \times t$ rectangle in $H$. We call the sum $\left| \sum_{i \in S} \sum_{j \in T} h_{ij} \right|$ the discrepancy of this rectangle.

Discrepancy measures the deviation from uniform distribution.

Exercise 2.3  Prove Lindsey’s Lemma.

Hint. Let $v_S \in \{0,1\}^n$ denote the incidence vector of $S \subseteq [n]$, i.e., the $(0,1)$-vector indicating membership in $S$. Observe that

$$\left| \sum_{i \in S} \sum_{j \in T} h_{ij} \right| = v_SHv_T^T.$$

Now use Exercise 1.8 and the Cauchy-Schwarz inequality:

$$\forall a, b \in \mathbb{R}^n \forall (a \cdot b) \leq ||a|| \cdot ||b||.$$

Definition 2.4  A rectangle is homogeneous if all of its entries are equal.

Exercise 2.5  If $H$ is an $n \times n$ Hadamard matrix, then $H$ has no homogeneous rectangles of area ($= st$) greater than $n$. 

3


**Exercise 2.6** (Erdős)
Prove: For all sufficiently large \(n\), \(\exists(n \times n) \pm 1\) matrices without homogeneous \(t \times t\) rectangles such that \(t \geq 1 + 2 \log_2 n\).

*Hint.* Use the Probabilistic Method. Flip a coin for each entry. Show that the probability that a random matrix is “bad” is less than 1. In fact it will be \(o(1)\) (almost all matrices are “good”).

**Exercise 2.7** Construct an explicit family of \((n \times n) \pm 1\) matrices \(A_n\) (for infinitely many values of \(n\)) such that \(A_n\) has no homogeneous \(t \times t\) rectangles for \(t > \sqrt{n}\).

**Open Problem 2.8** Construct an explicit family of \((n \times n) \pm 1\) matrices \(A_n\) (for infinitely many values of \(n\)) such that \(A_n\) has no homogeneous \(t \times t\) rectangles for \(t > n^{0.49}\).

3 Gale–Berlekamp Switching Game

Let \(A = (a_{i,j})\) be a matrix with entries \(\pm 1\). The first player sets the initial entries of \(A\). Subsequently the second player may switch any row or column (multiply the row or column by \(-1\)) and repeat this operation any number of times. The second player’s “score” is the quantity \(|\sum_{i,j \in [n]} a_{i,j}|\) which the second player wishes to maximize. The second player’s gain is the first player’s loss (zero-sum game), so the first player’s goal is to keep the second player’s score low.

Let \(m(n)\) denote the score an optimal Player 2 can achieve against an optimal Player 1.

**Exercise 3.1** Prove that \(m(n) = \Theta(n^{3/2})\).

*Hint 1.* \(m(n) = O(n^{3/2})\) requires Player 1 to be clever. Use an Hadamard matrix and Lindsey’s Lemma (Lemma 2.1). Warning: an \(n \times n\) Hadamard matrix may not exist (but a slightly larger one will be just as good).

*Hint 2.* \(m(n) = \Omega(n^{3/2})\). Player 2 needs a good strategy.

Let Player 2 flip a coin for each row to decide whether or not to switch that row. Subsequently, Player 2 should switch those columns whose sum is negative. Use the Central Limit Theorem for the analysis.