Definition 1. Let \( G \) be a finite abelian group, and let \( X \subseteq G \). We define
\[
\Phi(X) = \max_{\chi \neq \chi_0} \left| \hat{f}_X(\chi) \right| = \max_{\chi \neq \chi_0} \left| \sum_{g \in X} \chi(g) \right|
\]
where the maximum is taken over all non-principal characters \( \chi \in \hat{G} \setminus \{\chi_0\} \).
Recall that \( f_X \) denotes the characteristic function of \( X \):
\[
f_X(y) = \begin{cases} 1 & y \in X \\ 0 & y \notin X. \end{cases}
\]

Notation 2. Let \( X \subseteq G \). We will denote its complement by \( X = G \setminus X \). The set \( -X \) is defined by \( -X = \{ -x : x \in X \} \). Let \( a \in G \). The set \( X + a \) is defined by \( X + a = \{ x + a : x \in X \} \). Let \( k \in \mathbb{Z} \), \( x \in G \). The product \( kx \) is defined by
\[
kx = \begin{cases} x + \cdots + x \ (k \text{ times}) & \text{if } k > 0 \\ 0 \ (\text{in } G) & \text{if } k = 0 \\ -x + \cdots + -x \ (-k \text{ times}) & \text{if } k < 0. \end{cases}
\]
The set \( kX \) is defined by \( kX = \{ kx : x \in X \} \). Aut\((G)\) is the group of automorphisms of \( G \), i.e., the group of \( G \to G \) isomorphisms.

Exercise 3. Given \( X \subseteq G \), and \( |G| = n \), prove that
\[
\Phi(X) = \Phi(X) \\
\Phi(X) = \Phi(X + a) & a \in G \\
\Phi(X) = \Phi(-X) \\
\Phi(X) = \Phi(kX) & k \in \mathbb{Z}, \ \gcd(k,n) = 1 \\
\Phi(X) = \Phi(\alpha(X)) & \text{for } \alpha \in \text{Aut}(G).
\]

Exercise 4. Prove
\[
|\text{Aut}(\mathbb{Z}_m)| = \varphi(m)
\]
where \( \varphi \) is Euler’s phi function.

Exercise 5. (Sets with largest non-principal Fourier coefficient)
Let \( a_{n,k} = \max\{ \Phi(X) : X \subseteq \mathbb{Z}_n, |X| = k \} \). Let \( a_n = \max_k a_{n,k} \).
(a) Prove: if \( n \) is a prime and \( \Phi(X) = a_{n,k} \), where \( X \subseteq \mathbb{Z}_n \) and \( |X| = k \), then \( X \) is an arithmetic progression \((\mod n)\) of length \( k \).
(b) Prove: (a) is false if \( n \) is composite. \( \text{Hint: let } k = 2. \)
(c) Prove: \( a_p \) is an increasing function of \( p \) for primes \( p \).

(OVER)
(d) Prove: \( a_p \sim p/\pi \) for primes \( p \) as \( p \to \infty \).
(e) Prove the following unusual characterization of \( \pi \):

\[
\pi = \sup_{G} \min_{X} \frac{|G|}{\Phi(X)},
\]

where \( G \) runs over all finite abelian groups, and \( X \) runs over all subsets of \( G \).