Study-guide problems for midterm

1. Solve the initial-value problems
   \[ y' = -y + x, \quad y(0) = 1, \]
   \[ y' = y + x, \quad y(0) = 1. \]

2. Find the general solution of the equation
   \[ y' = ay + b \cos kt \]
   where \( a, b \) and \( k \) are any nonzero real constants.

3. Solve the initial-value problem
   \[ y' = xe^{y-x^2}, \quad y(0) = 0. \]

4. Which of the following sets of functions are linearly independent on \( \mathbb{R} \):
   (a) \( 1, x, x^2, x^4 \).
   (b) \( \sin x, \cos x, \sin(x + 1) \).
   (c) \( e^x - e^{-2x} + e^{-x}, e^{-2x} + e^x, \cosh x \).

5. Let \( f(x) \) be a \( C^1 \) function on an interval \( I \) of the \( x \)-axis. Give sufficient conditions for the pair \( f(x), xf(x) \) to be linearly independent on \( I \).

6. Which of the following pairs of functions could be a basis of solutions for a differential equation \( u'' + p(x)u' + q(x)u = 0 \) with continuous coefficients on the real axis?
   (a) \( \sinh x, x \sinh x \).
   (b) \( \sinh x, \cosh x \).
7. Find real bases of solutions for the following equations:

(a) \( u'' + u' + u = 0 \).

(b) \( u'' + u' - u = 0 \).

(c) \( u''' + 2u'' - 2u' - 4u = 0 \) (Hint: \( \exp(-2x) \) is a solution.)

8. Put each of the equations of the preceding problem in system form, i.e.
\[
v' = Av,
\]
i.e., find the matrix \( A \) in each case.

9. Find a basis of solutions for the equation
\[
(D^2 + D + 1)^2 u = 0.
\]

10. Consider the equation
\[
u'' - \frac{2}{(1 + x)^2} v = 0.
\]
Verify that \( v = (1 + x)^2 \) is a solution, and find a second, linearly independent solution.

11. The motion of a pendulum near its rest state (\( \theta = 0 \)) is described by the equation
\[
L\theta \equiv \ddot{\theta} + \nu \dot{\theta} + \omega^2 \theta = r(t),
\]
where \( 0 < \nu < \omega \). Here \( \nu \) is a coefficient of friction, and \( r \) represents a forcing term. Find the general solution if \( r(t) = \cos(kt) \).

12. Solve the initial-value problem
\[
u'' + \frac{2x}{1 - x^2} u' - \frac{2}{1 - x^2} u = 0, \quad u(0) = 1, \ u'(0) = 0.
\]
Hint: \( u(x) = x \) is a solution of the differential equation.
13. Let

\[ Lu = u'' + p(x)u' + q(x)u \]  \hspace{1cm} (1)

with continuous coefficients \( p, q \) on an interval \([a, b]\). Show that if \( u \) and \( v \) are linearly independent solutions of \( Lu = 0 \) and \( u \) has consecutive zeros \( x_1 \) and \( x_2 \), then \( v \) has a zero in the interval \((x_1, x_2)\).

14. (a) For the inhomogeneous equation \( Lu = r(x) \) where \( L \) is defined above (equation (1)) and \( r \) is a continuous function on \([a, b]\), explain how to obtain the influence function \( G(x, \xi) \) providing a particular integral in the form

\[ u_P(x) = \int_a^x G(x, \xi)r(\xi) \, d\xi. \]

(b) Find an explicit influence function providing a particular integral of the differential equation

\[ u'' + k^2 u = r(x), \quad x \in \mathbb{R} \]

where \( k \) is a real, nonzero constant and \( r \) is an arbitrary continuous function. Express the most general solution of this equation.

15. State without calculation the minimum-possible radius of convergence of power-series solutions of

\[ w'' - (1 - z)^{-1}w' - \frac{1}{4}(1 - z)^{-2}w = 0. \]

On what theorem do you rely? Find a linearly independent pair of power series solutions of this equation.

16. Airy's equation is

\[ w'' + zw = 0. \]

Find the power-series solution with initial data \( w(0) = 1, w'(0) = 0 \). What is its radius of convergence?

17. Consider the initial-value problem

\[ w' = w^2, \quad w(0) = 1. \]

Find the first four terms of the power-series expansion. Compare with the exact solution.