Solutions for Problem Set 4

Problem Set 3.6.1, no.1

\[ |A - \lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + \cdots + a_n), \]
i.e., except for a factor \((-1)^n\), it is the characteristic polynomial.

Problem Set 3.6.1, no.3

\[ e^\sigma = \cos(1)I + \sin(1)\sigma. \]

Problem Set 3.6.1, no.4 Put \(x_1 = -\omega u, \ x_2 = \dot{u}.\)

Problem Set 3.6.1, no.6 Put \(A = \lambda I + Z\) where

\[ Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

Then (induction) \(A^n = \lambda^n I + n\lambda^{n-1}Z\) and one finds

\[ \exp A = \sum_{k=0}^{\infty} \frac{1}{n!} A^n = e^\lambda (I + Z) = e^\lambda \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

Problem Set 3.6.1, no.7

\[ x_1 = e^{\lambda t} - te^{\lambda t}, \ x_2 = -e^{\lambda t}. \]

Problem Set 3.6.1, no.8 There are various ways to approach this. Finding, as in problem no. 6 above, the fundamental matrix solution \(e^{At} = \Phi(t)\) one finds

\[ \Phi(t) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \]

and therefore \(\Phi^{-1}(s) = \begin{pmatrix} e^{-\lambda s} & -se^{-\lambda s} \\ 0 & e^{-\lambda s} \end{pmatrix} \)

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so the solution wanted is
\[ x(t) = \int_0^t \Phi(t)\Phi^{-1}(s)R(s) \, ds. \]

This works out, after some elementary integrations, to be
\[ \frac{e^{\lambda t} - 1}{\lambda} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}. \]

Problem Set 3.6.1, no.9
First consider \( x = (1, 0, \ldots, 0)^t \equiv e_1 \), the first unit vector. It has \( \|e_1\| = 1 \).
Then \( Ae_1 = (a_{11}, a_{21}, \ldots, a_{n1})^t \) with \( \|Ae_1\| = \sum_{i=1}^{n} |a_{i1}| \) so \( \|A\| \) is at least this big. Repeating this for \( x = (0, 1, 0, \ldots, 0) = e_2 \) etc. we see that
\[ \|A\| \geq \sup_{j=1,2,\ldots,n} \sum_{i=1}^{n} |a_{ij}|. \] (1)

On the other hand, for any vector \( x \) we have \( x = x_1e_1 + \cdots + x_ne_n \) so
\[ Ax = x_1Ae_1 + \cdots + x_nAe_n \] so \( \|Ax\| \leq \sup_j \|Ae_j\| \|x\|, \)
so for \( \|x\| = 1 \), we see that \( \|A\| \leq \sup_{j=1,2,\ldots,n} \sum_{i=1}^{n} |a_{ij}|. \) Together with equation (1), this proves the claim.

Supplementary Problems for Chapter 3

1. Let \( A \) be the \( n \times n \) companion matrix as in equation (3.65) of the text. Suppose \( \lambda \) is an eigenvalue of multiplicity two and let \( \xi \) be an eigenvector: \( A\xi = \lambda \xi \). You may regard as established that there is a second, linearly independent, vector \( \eta \) such that \( A\eta = \lambda \eta + \xi \). Show that the time-dependent function \( x(t) = \exp(\lambda t)\xi \) is a solution of the equation \( \dot{x} = Ax \). Find a second, linearly independent solution of this equation as a linear combination of \( \xi \) and \( \eta \) with coefficients depending on \( t \).

A second solution is
\[ e^{\lambda t} (\eta + t\xi) \]
as can be verified by direct calculation.
2. Consider the system of 3 equations

\[ \frac{dy}{dt} = Ay \]

where

\[ A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}. \]

Find the fundamental matrix solution reducing to the identity when \( t = 0 \). Express it as a linear combination of the identity \( I \) and of \( A \) itself, with coefficients that depend on \( t \).

Observe that \( A^2 = 6A \). The fundamental matrix solution is

\[
\exp(At) = I + At + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots \\
= I + A \left( t \frac{6t^2}{2!} + \cdots \frac{6^{n-1}t^n}{n!} + \cdots \right) \\
= I + \frac{1}{6}A \left( 6t + \frac{(6t)^2}{2!} + \cdots + \frac{(6t)^n}{n!} + \cdots \right) \\
= I + \left( e^{6t} - 1 \right) A
\]

where the last line gives the required expression.

3. Let \( A \) be the matrix of problem 6 in Problem Set 3.6.1. Suppose that \( \lambda < 0 \) and consider the equation \( \dot{x} = Ax \).

(a) Show that all solutions decay to zero as \( t \to \infty \).

The general solution is

\[ x_1(t) = c_1 e^{\lambda t} + c_2 te^{\lambda t}, \quad x_2(t) = c_2 e^{\lambda t}. \]

These tend to zero as \( t \to \infty \) if \( \lambda < 0 \).

(b) Define \( \|y\| = \sqrt{y_1^2 + y_2^2} \). Show that there exist initial data \( y(0) \) such that, if \( |\lambda| \) is small enough, the maximum value attained by the ratio \( \|y(t)\|/\|y(0)\| \) can be made as large as we please.

Choose \( c_1 = y_1(0) = 0 \) and put \( c_2 = y_2(0) = a > 0 \) so that \( \|y(0)\| = a \). Then \( \|y(t)\|^2 = a^2 e^{-2|\lambda| t} (1 + t^2) \). Differentiating this
and setting the derivative equal to zero to find the extreme points, one finds them at

\[ t_1 = \frac{1 - \sqrt{1 - \lambda^2}}{|\lambda|} \quad \text{and} \quad t_2 = t_1 = \frac{1 + \sqrt{1 - \lambda^2}}{|\lambda|}. \]

For small \(|\lambda|\) one finds \(t_1 \approx |\lambda|/2\) corresponding to a local minimum and \(t_2 \approx 2/|\lambda|\) corresponding to a local maximum. The value of \(\|y(t_2)\|^2 \approx 4a^2/e^2|\lambda|^2\) so the ratio

\[ \frac{\|y(t_2)\|}{\|y(0)\|} \approx \frac{2}{e|\lambda|}, \]

which becomes arbitrarily large as \(|\lambda|\) tends to zero. The maximum ratio of \(\|y(t)\|\) to \(\|y(0)\|\) may not occur for the choice made of \(y(0)\), but it can only be greater than the value found.