From Problem Set 6.2.1

1. 2) The equivalent integral equation is

\[ y(x) = 1 + \lambda \int_0^x y(s) \, ds. \]

The first three approximations are

\[ y_0 = 1, \quad y_1 = 1 + \lambda x, \quad y_2 = 1 + \lambda x + (1/2)\lambda x^2. \]

The exact solution is

\[ y = \exp(\lambda x) = 1 + \lambda x + (1/2)\lambda x^2 + \cdots \]

2. 4) From the equivalent differential equation, \( u(x) = \exp(x^3/3) \).

3. 7) The solution is

\[ u(x) = \frac{1}{(x + 3/2)(5/2 - x)} \]

with maximal interval \((-3/2, +5/2)\).

4. 8) Differentiating each side of the \( u' \) equation and using the \( v' \) equation gives an equation relating \( u'' \) to \( u', u, \) and \( v; \) a second use of the original \( u' \) equation then eliminates \( v, \) giving

\[ u'' + p(x)u' + q(x)u = 0, \]

where

\[ p(x) = -(a + d + b'/b), \quad q(x) = -(a' + bc - ad - ab'/b). \]

In addition to the differentiability requirements, it is necessary that \( b(x) \) not vanish on \( I. \)

5. 12) By definition

\[ \left| \int_a^b U(x) \, dx \right| = \sum_{i=1}^n \left| \int_a^b u_i(x) \, dx \right|. \]

Since \( \left| \int u_i(x) \, dx \right| \leq \int |u_i(x)| \, dx \) the right-hand side above is not greater than \( \int_a^b \sum_{i=1}^n |u_i(x)| \, dx, \) and the latter is \( \int_a^b |U(x)| \, dx. \)
1. Consider the initial-value problem

\[ y' = f(x, y), \quad y(x_0) = y_0 \]

where \( y \in \mathbb{R}^n \) and \( f \) is defined in a domain \( D \subset \mathbb{R}^{n+1} \) and satisfies the conditions of Theorem 6.2.1. Denote by \( x = b \) the right-hand endpoint of the maximal interval of existence of the solution \( y \).

Prove the following

**Theorem 0.1** Suppose the vector field \( f(x, y) \) is bounded on the domain \( D \). Then the graph \( (x, y(x)) \) of the solution tends to a particular point of \( \partial D \).

**SOLUTION** The integral form of the solution is

\[ y(x) = y_0 + \int_{x_0}^{x} f(s, y(s)) \, ds. \]

Let \( x_n \) be any sequence converging to \( b \) from the left (i.e., \( x_n < b \)), and put \( y_n = y(x_n) \). By the equation above

\[ y_n - y_m = \int_{x_m}^{x_n} f(s, y(s)) \, ds. \]

Since \( f \) is bounded (by \( M \) say) it follows that

\[ |y_n - y_m| \leq M|x_n - x_m|. \]

The convergence of the sequence \( x_n \) therefore implies the convergence of the sequence \( y(x_n) \). It is a theorem of analysis that if the latter sequence is convergent for any sequence \( x_n \) converging to \( b \), then the limit of \( y(x_n) \) is the same (call it \( c \)) for any choice of the \( x_n \) and that the function \( y(x) \) has this as its limit as \( x \to b \). Thus \( (x, y(x)) \to (b, c) \).

2. In one dimension, consider

\[ \dot{y} = y^{-1} t, \quad y(0) = y_0. \]

Assume that \( y_0 > 0 \) and take \( D = \{(t, y) : t \in \mathbb{R}, \quad y > 0\} \). Find the solution explicitly. Does its graph approach \( \partial D \)? What is its maximal interval of existence?
SOLUTION The solution is \( y(t) = \sqrt{y^2_0 + t^2} \). \( \partial D \) is the line \( y = 0, t \in \mathbb{R} \) and the solution does not approach the boundary. Its maximal interval of existence is \((-\infty, +\infty)\).

3. In one dimension, consider

\[
\dot{x} = -t^{-2} \cos(1/t), \quad x(1/\pi) = 0.
\]

For the domain take \( D = \{x \in \mathbb{R}, \quad t > 0\} \). Find the solution explicitly. Show that it approaches the boundary of \( D \) but does not approach a particular point of the boundary.

SOLUTION The solution is \( x(t) = \sin(1/t) \). The boundary is \( t = 0 \) with \( x \) unrestricted. As \( t \to 0 \), \( x(t) \) oscillates increasingly rapidly and takes on all \( x \) values between \(-1\) and \(+1\), thus approaching the boundary without approaching a particular point of the boundary.

4. In two dimensions, consider

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= 2(t - 1)^{-2}x_1; \quad x_1(0) = 1, \quad x_2(0) = -2.
\end{align*}
\]

For the domain take \( D = \{x \in \mathbb{R}^2, \quad t < 1\} \). Find the solution explicitly and show that, even though the vector field is unbounded on \( D \), the solution approaches a particular point of the boundary as \( t \to 1 \).

SOLUTION A linearly independent pair of solutions is

\[
\begin{align*}
x^{(1)}(t) &= \begin{pmatrix} (1 - t)^2 \\ -2(1 - t) \end{pmatrix}, \\
x^{(2)}(t) &= \begin{pmatrix} (1 - t)^{-1} \\ -(1 - t)^{-2} \end{pmatrix}.
\end{align*}
\]

This can be checked directly and could be discovered by the following steps: (1) observe that \( d^2x_1/\!dt^2 - 2(1-t)^{-2}x_1 = 0 \) and (2) that the transformation \( \tau = \ln(1 - t) \) converts the equation to \( d^2x_1/\!d\tau^2 + dx_1/\!d\tau - 2x_1 = 0 \), which can be solved easily and, on reconverting from \( \tau \) to \( t \), gives the solutions above.

With the given initial data the solution is \( x^{(1)}(t) \), which tends to the boundary point \((x_1, x_2, t) = (0, 0, 1)\) as \( t \to 1 \).

5. In two dimensions, consider the Hamiltonian system

\[
\begin{align*}
\dot{x}_1 &= \frac{\partial H}{\partial x_2}, \\
\dot{x}_2 &= -\frac{\partial H}{\partial x_1},
\end{align*}
\]
where \( H(x_1, x_2) \) is a \( C^2 \) function defined on all of \( \mathbb{R}^2 \) and satisfies the condition \( H \to \infty \) as \( x_1^2 + x_2^2 \to \infty \). Show that, for arbitrary initial data, the solution exists for all \( t \in \mathbb{R} \).

**SOLUTION** For this system of equations, the domain \( D \) is unbounded so (Theorem 6.1.5) either the right-hand endpoint \( b \) is at \( \infty \) or the solution \( x(t) \) is unbounded as \( t \to b \). The function \( H(x_1, x_2) \) is constant on solutions and therefore \( H(x_1(t), x_2(t)) \) is equal to its initial value \( H(x_1(0), x_2(0)) \). If, as \( t \to b \) \( x(t) \) were unbounded, then \( H(x_1(t), x_2(t)) \) would also have to be unbounded. Since it is constant, it must be that \( b = +\infty \). Similar reasoning shows that the left-hand endpoint is \( a = -\infty \).