Kick-Ass

CALCULUS

An Introduction

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1.1 What is a Limit?

Limits form the foundations of calculus: without limits, we would not be able
to even conceive of the derivative or integral; all of calculus rests on this single
concept. Unfortunately, the limit is one of the most difficult concepts for
students of mathematics to grasp—the relevant definitions often seem arbitrary,
and largely unrelated to what we might think a limit should be. At the very
least, the oft-taught “ε-δ” definition of a limit often seems to be needlessly
complex. Why must a concept which to some seems so intuitively simple be
formulated in such an obtuse, seemingly arbitrary definition? We will begin
this chapter by exploring the concept of a limit and developing a definition
which will coincide with our intuition of what the limit of a function at a point
should be. Section 1.2 will develop the machinery necessary to prove that a
point is a limit of a function, and Section 1.3 will move on to more practical
matters, developing the theoretical basis for the actual computation of limits.
Finally, we will discuss the left and right-hand limits, and address the concept
of infinite limits of both sorts: limits which have values defined to be infinite,
and limits of functions “as x goes to infinity.”

1.1.1 Our Intuition Regarding Limits

Consider the the function $g(x) = \frac{x}{2}$. $g(x)$ is just a line, and at any, point, $g(x)$
takes half the value of that point. So $g(1) = \frac{1}{2}$, $g(0) = 0$, etc. and $g(x)$ isn’t
very interesting. So let’s look at another function, $f(x)$, defined as follows:
1. Limits

Figure 1.1: The graph of \( f(x) = \frac{x(x-1)}{2(x-1)} \); this function isn’t defined at \( x = 1 \), but it certainly seems to get really close to the value \( \frac{1}{2} \) there

\[ f(x) = \frac{x(x-1)}{2(x-1)} \]

(see Figure 1.1)

It is clear that for most \( x \) values, this function gives the same value as the simple line did. \( f(5) \), for example, gives \( \frac{5 \cdot 4}{2 \cdot 4} = \frac{5}{2} \), and \( f(7) \) gives \( \frac{7 \cdot 6}{2 \cdot 6} = \frac{7}{2} \). In fact, it appears that \( f(x) \) might be the same as \( g(x) \) for all \( x \), but this is not so: for \( x = 1 \), \( f(x) = \frac{1 \cdot 0}{2 \cdot 0} = \frac{0}{0} \), which is undefined.

Is this the only value for which \( f(x) \) is different from the line \( g(x) \)? Yes: For all \( x \) other than 1, the \((x - 1)\) terms on the top and bottom of the fraction \( \frac{x(x-1)}{2(x-1)} \) will cancel, leaving only \( \frac{x}{2} \), just as with \( g(x) \). So the graph of \( f(x) \) is like the graph of \( g(x) \) with the point \((1, \frac{1}{2})\) removed, as shown in figure 1.1.

For our function \( f(x) \), there is no value at \( x = 1 \). Yet it seems like the point \((1, \frac{1}{2})\) still is a significant point in the graph of the function. In trying to describe the relation between \( f(x) \) and this point \((1, \frac{1}{2})\), we might say that as \( x \) gets closer and closer to \( \frac{1}{2} \), \( f(x) \) approaches 1. This seems to hold true: \( f(.9) = .45 \), \( f(1.1) = .55 \), \( f(.99) = .495 \), etc. We call \( \frac{1}{2} \) the limit of \( f(x) \), and we aim to discover a definition of this concept of a limit which will coincide with our intuitions about what values a function approaches.

1.1.2 How Close Can You Go?

In light of the previous example, let’s try to define a limit of a function as follows:

**Attempted Definition of the Limit** 1. \( \lim_{x \to c} f(x) = l \) if \( f(x) \) gets closer and closer to \( l \) as \( x \) gets closer and closer to \( c \)

*Note.* This is read as “The limit as \( x \) approaches \( c \) of \( f(x) \) is \( l \) if \( f(x) \) gets closer and closer to \( l \) as \( x \) gets closer and closer to \( c \).”

Does this definition coincide with our intuition of what a limit should be?
What is a Limit?

![Figure 1.2: $h(x) = x^2$: We would expect the limit of this function to be 0 as $x$ approaches 0.](image)

Well, it certainly seems to work with our last example: We wanted the limit of $f(x)$ to be $\frac{1}{2}$ as $x$ approaches 1, and $f(x)$ certainly seems to get closer and closer to $\frac{1}{2}$ as $x$ gets closer and closer to 1. Then, according to this definition, we have

$$
\lim_{x \to 1} f(x) = \frac{1}{2}
$$
as desired.

But will this definition always correspond with our intuition of a limit? Let’s define another function, $h(x)$ as

$$
h(x) = x^2
$$
(see Figure 1.2).

In some ways, $h(x)$ looks a lot nicer than $f(x)$ did. It doesn’t seem like there are any “missing” points to $h(x)$. Looking at the graph of $h(x)$, it doesn’t seem that we should expect it to “approach” any points other than points in the graph: as $x$ approaches 0, for example, we would like to expect the limit of $h(x)$ to be 0 in turn. Is this what our definition from above implies?

Well, it certainly seems to work, at least at first. We can observe that as $x$ gets closer and closer to 0, $f(x)$ seems to get closer and closer to 0 as well. So, once again, we have what we want: in this case, $\lim_{x \to 0} h(x) = 0$.

But wait a second, doesn’t $h(x)$ get closer and closer to other values as well? It certainly looks like $h(x)$ gets closer and closer to $-\frac{1}{2}$ as $x$ gets closer and closer to 0 (marked with an x on the graph). But this means that $\lim_{x \to 0} h(x) = -\frac{1}{2}$ is also true! As well as $\lim_{x \to 0} h(x) = -1$, and $\lim_{x \to 0} h(x) = -2$, and so on and so forth. Where is our definition going wrong?

Why don’t we want these “other” values to be limits of the function $h(x)$? Intuitively, they don’t seem close enough to the function. Sure, $h(x)$ gets closer and closer to $-1$ as $x \to 0$, but it doesn’t seem to get very close: $h(x)$ is always at least 1 away from $-1$. But how can we fix our definition? We can’t say that $h(x)$ would just have to get “really close” to a number $l$.
for it to be the limit. How would we decide if it was really close? \( h(x) \) gets really close to \(-0.1\) as \( x \to 0 \) by some standards, but we still wouldn’t want \( \lim_{x \to 0} h(x) = -0.1 \) to be true. But then how close do we think \( f(x) \) needs to get to \( l \) for it to really be the limit?

1.1.3 As Close as You Want...

To understand how \( h(x) \) gets close to 0 as \( x \) approaches 0 in a way that it doesn’t get close to, say, \(-\frac{1}{2}\), we must realize that \( h(x) \) gets arbitrarily close to 0. That is, \( h(x) \) gets as close as you want to 0 as \( x \) approaches 0. Is this true for another value, such as \(-1\)? Clearly not. As \( x \to 0 \), \( h(x) \) is always at least 1 away from \(-1\). This problem will arise with any point less than 0. In the case of the \( l = -1 \), for example, \( h(x) \) never gets closer than being .1 away from \( l \), and so this \( l \) cannot be the limit.

Is this sufficient then to describe what we mean by a limit? Provisionally, let’s redefine the limit with this concept of \( h(x) \) getting arbitrarily close to \( l \).

For convenience, we will denote as \( \varepsilon \) (the Greek letter \( \varepsilon \)-sigma) any distance we want a function \( f(x) \) to be closer than to its limit \( l \). Then we can say that \( f(x) \) gets as close as we want to its limit \( l \) if \( f(x) \) gets closer than \( \varepsilon \) to \( l \) no matter what value we choose for \( \varepsilon \), or in other words, for any \( \varepsilon \).

**Attempted Definition of the Limit 2.** \( \lim_{x \to c} f(x) = l \) if for any \( \varepsilon \), \( f(x) \) gets closer than \( \varepsilon \) to \( l \) for values of \( x \) getting closer to \( c \)

*Note.* This can be read as “\( l \) is the limit of \( f(x) \) as \( x \) goes to \( c \) if \( f(x) \) gets arbitrarily close to \( l \) for values of \( x \) getting closer to \( c \)”.

We must be careful what we mean, now, by our requirement that \( h(x) \) gets arbitrarily close to \( l \). In applying this definition to determine if 0 was the limit of \( h(x) \) as \( x \to 0 \), for example, we would try to determine if we could pick arbitrarily small distances (i.e. \( \varepsilon \)), and find that the the value of \( h(x) \) would at some point for \( x \) get closer than \( \varepsilon \) to the point \( l \). But do we care if the function stays that close as \( x \) gets even closer to \( c \)? If not, we run into problems with functions like that shown in figure 1.3. It doesn’t look like this function should have a limit at 0, but it does according to our previous definitions. The function shown in figure 1.3 clearly gets as close as we want to the value 0, it just doesn’t stay there! In fact, the function gets as close as you want to 1 as well, and all the values in between!

How can we fix our definition so that we won’t get a limit for a function like this? For the limit of \( f(x) \) to be \( l \) as \( x \) approaches \( c \), it seems we need to require that no matter how close we want \( f(x) \) to get to \( l \), there is a point as \( x \) approaches \( c \) at which \( f(x) \) gets that close to \( l \) and then stays that close as \( x \) continues to approach \( c \).
What is a Limit?

Figure 1.3: The graph of \( y = \sin \frac{1}{x} \), oscillating back and forth between \(-1\) and \(1\). What is the limit as \( x \to 0 \) of this function?

Figure 1.4: Visualizing the limit of \( f(x) = \frac{x(x-1)}{2(x-1)} \): We know the limit must be \( \frac{1}{2} \) at 1 because no matter what \( \varepsilon \) we want our \( f(x) \) to be within of \( \frac{1}{2} \), we can find values \( 1 - \delta \) and \( 1 + \delta \) such that \( f(x) \) is that close for all \( x \) that are closer to 1 than those values are.

In other words, if we denote as \( \varepsilon \) the arbitrary distance we want \( f(x) \) to be within \( l \) of, then \( l \) is the limit as \( x \) approaches \( c \) of \( f(x) \) if there is some point such that for all values of \( x \) closer to \( c \) than this point, \( f(x) \) is at least within \( \varepsilon \) of \( l \). The application of this idea of the limit to the first function from the section is shown in figure 1.4. Essentially, though, we have arrived at an acceptable definition of a limit of \( f(x) \):

**Definition of the Limit 1.**

\[
\lim_{x \to c} f(x) = l
\]

if, for any \( \varepsilon \), there is a region around the point \( c \) such that for all \( x \) in that region, \( f(x) \) is within \( \varepsilon \) of \( l \).

We will now try to write our definition of the limit with mathematical symbols, replacing our vocabulary for distances and regions with mathematical “sentences” that have the same meaning. To do this, we need to be able to mathematically write statements such as “\( f(x) \) is within \( \varepsilon \) of \( c \)” and “when \( x \) is in a region around \( c \)”. To this end, consider the mathematical statement

\[
|f(x) - l| < \varepsilon
\]

How can we read this statement? It says that the absolute value of the
difference between \( f(x) \) and \( l \) is less than \( \varepsilon \). In other words, the distance between \( f(x) \) and \( l \) is less than \( \varepsilon \). In our definition, when do we want this to be a true statement? We need this distance between \( f(x) \) and \( l \) to be less than \( \varepsilon \) whenever \( x \) is in some specific region around \( c \). How can we write that \( x \) is in some region around \( c \)? If \( x \) is in a certain region around \( c \), then it is within a certain distance of \( c \). But this means that we can write this using the same notation as above. Denoting as \( \delta \) (the Greek letter delta) the distance that \( x \) is within \( c \) of, we can write

\[
0 < |x - c| < \delta
\]

to mean “\( x \) is in a region around \( c \) of radius \( \delta \).” (See figure 1.5)

Notice that, unlike the region for \( f(x) \) around \( l \), we don’t want this region to include the point \( c \) that it’s centered around (that’s what the “\( 0 < |x - c| \)” part of the inequality means, because \( x = c \) only when their distance is 0). We don’t want the region to include \( c \) because we don’t want the limit of \( f(x) \) at \( c \) to depend on what the function’s value is at \( x = c \), but instead only what the function’s values are as \( x \) gets close to \( c \). Remember, our first function \( f(x) \) wasn’t even defined at \( c \) (let alone close to \( l \) at \( c \)), but we still wanted the limit of \( f(x) \) to exist.

To summarize, we now know

\[
|f(x) - l| < \varepsilon
\]

to be the same as the statement that “\( f(x) \) is within a region of size \( \varepsilon \) around \( l \),” and

\[
0 < |x - c| < \delta
\]

to be the same as the statement that “\( x \) is in a region of width \( \delta \) around \( c \), not including \( c \).”

For these to make sense, we need \( \varepsilon \) and \( \delta \) to be greater than 0, as the notion of a negative distance is meaningless in this case.

Armed with this notation for mathematically denoting intervals around the values \( c \) and \( l \), we are now ready to rewrite our definition for the limit. Replacing vocabulary in definition 1 with these mathematical descriptions of the intervals, we get
Definition of the Limit 2.

\[ \lim_{{x \to c}} f(x) = l \]

if, for any \( \varepsilon > 0 \), there is some \( \delta > 0 \) such that when

\[ 0 < |x - c| < \delta \]

then

\[ |f(x) - l| < \varepsilon \]

How is this the same as the statement of definition 1? Look at table 1.1 to help understand what this definition says. Is it clear that definition has the same meaning as definition 1?

<table>
<thead>
<tr>
<th>Mathematical Definition</th>
<th>Intuitive Meaning</th>
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<tr>
<td>( \lim_{{x \to c}} f(x) = l ) if...</td>
<td>The limit of ( f(x) ) as ( x ) approaches ( c ) is ( l ) if...</td>
</tr>
<tr>
<td>For any ( \varepsilon &gt; 0 )</td>
<td>for any distance ( \varepsilon ) you want ( f(x) ) to be closer to its ( l ) than,</td>
</tr>
<tr>
<td>There exists some ( \delta ) such that</td>
<td>There is some distance ( \delta ) such that</td>
</tr>
<tr>
<td>If ( 0 &lt;</td>
<td>x - c</td>
</tr>
<tr>
<td>(</td>
<td>f(x) - l</td>
</tr>
</tbody>
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Table 1.1: The meaning of the mathematically stated definition of the limit (Definition 2)

1.1.4 Applying the Definition

So what can we do with the definition of the limit? The definition states that “\( \lim_{{x \to c}} f(x) = l \) if...“ So the definition tells us whether or not a point (i.e. \( l \)) is the limit of \( f(x) \) at \( c \). Well let’s try it out!

Consider the function

\[ f(x) = \begin{cases} \frac{1}{2}x & \text{when } x \neq 2 \\ 2 & \text{when } x = 2 \end{cases} \]

(see figure 1.6)
1. Limits

What is it’s limit as \( x \) approaches 2? Our definition of a limit, you’ll notice, is not a formula for telling us what the limit is, but only a way of checking whether or not what we think the limit is actually is the limit. The definition is not worded “\( \lim_{x \to c} f(x) = \ldots \)” but instead as “\( \lim_{x \to c} f(x) \) is \( l \) if \ldots .” So to apply the definition, we need to hypothesize that some \( l \) is the limit of \( f(x) \) at \( c \). Looking at the graph of the function, it looks like 1 would be a good value to try.

How do we tell if 1 is the limit of \( f(x) \) at 2? According to our definition, we need know that for any \( \varepsilon \), there is some \( \delta \), such that as long as \( x \) is within \( \delta \) of 2, \( f(x) \) is within \( \varepsilon \) of 1.

This seems difficult to know. How can we know that we’ll be able to find a \( \delta \) like this no matter what the \( \varepsilon \)? One \( \delta \) won’t work for any \( \varepsilon \) in this case—How close we need \( x \) to be to 2 will depend on how close we’re trying to make \( f(x) \) to \( l \).

To be sure that we’ll always be able to have a \( \delta \) that satisfies our definition, therefore, we won’t find a specific \( \delta \), but instead, we’ll find a way of finding a \( \delta \) for any given \( \varepsilon \).

So, what about our \( f(x) \)? Well, first let’s look at some specific \( \varepsilon \) and see if we can find a \( \delta \) for each one. This won’t prove that we’ll always be able to find one, but it will give us a feel for the problem.

What about for \( \varepsilon = \frac{1}{2} \)? For what region of \( \delta \) around 2 will \( f(x) \) always be that close to 1? Well, \( f(x) \) is within \( \frac{1}{2} \) of 1 when it’s between \( \frac{1}{2} \) and \( 1 \frac{1}{2} \). So we need only to know for what values of \( x \) it is within this region. Of course, \( f(x) \) isn’t in this region at \( x = 2 \) (see Figure 1.7) but this is irrelevant. Remember, the region around \( c \) (in this case \( c \) is 2) defined by \( 0 < |x - c| < \delta \) does not include the point c. So, whenever we’re talking about the region around 2 for \( x \) for which \( f(x) \) will be within \( \varepsilon \) of \( l \), we don’t care what \( f(x) \) is when \( x \) actually is \( c \), but only when it is in that region around \( c \).

Taking this into account, we notice that for all values of \( x \) other than \( c \), \( f(x) = \frac{1}{2} x \). So when is \( \frac{1}{2} x \) between \( \frac{1}{2} \) and \( 1 \frac{1}{2} \)? When \( x \) is between 1 and 3, of
course. For what \( \delta \) is the region from \( 2 - \delta \) to \( 2 + \delta \) equal to the region from 1 to 3? For \( \delta = 1 \). So we can say that for \( \varepsilon = 3 \), we can take \( \delta = 1 \), and then whenever \( 0 < |x - 2| < \delta \), we have \( |f(x) - 1| < \varepsilon \). Of course, we haven’t proven anything yet about that limit—we’ll need to show that we can do this for any \( \varepsilon \) to do that.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>We need ( \delta = )</th>
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<tr>
<td>( \frac{1}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{4} )</td>
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Table 1.2: \( \delta \)'s for \( \varepsilon \)'s

But what about trying another specific \( \varepsilon \); say, \( \varepsilon = \frac{1}{4} \)? Can we find a \( \delta \) that makes \( f(x) \) always be this close to 1? Before, we needed to make \( f(x) \) between \( \frac{1}{2} \) and \( 1 \frac{1}{2} \), but now we need to make it between \( \frac{3}{4} \) and \( 1 \frac{1}{4} \). This will be true when \( x \) is between \( 1 \frac{1}{2} \) and \( 2 \frac{1}{2} \) (Remember, for \( x \neq 2 \), \( f(x) = \frac{1}{2}x \), so when \( x \) is within \( \delta = \frac{1}{2} \) of 2, \( f(x) \) is within \( \varepsilon = \frac{1}{4} \) of 1.

Let’s try one more value for \( \varepsilon \); \( \varepsilon = \frac{1}{8} \). Here we’ll need \( f(x) \) to be between \( \frac{7}{8} \) and \( 1 \frac{3}{4} \), so we’ll need \( x \) to be between \( 1 \frac{1}{2} \) and \( 2 \frac{1}{2} \). What’s our \( \delta \)? We need \( \delta = \frac{1}{4} \). Our results are summarized in Table 1.2.

Our results seem to suggest a pattern—no matter what \( \varepsilon \) we’re given, taking \( \delta \) to be twice its value seems to work. Will this always be the case? To find out, let’s try doing the same problem without a specific \( \varepsilon \) in mind. Assume we’re given some \( \varepsilon \). Will taking \( \delta = 2\varepsilon \) ensure that when \( x \) is within \( \delta \) of 2, \( f(x) \) is within \( \varepsilon \) of 1?

\[ f(x) = \frac{1}{2}x \text{ for } x \neq 2 \] (and remember, we don’t care what \( x \) is at 2). And if
we’ve taken \( \delta = 2\varepsilon \), then when \( x \) is between \( 2-\delta \) and \( 2+\delta \) (i.e. \( 0 < |x-2| < \delta \)), \( x \) must be between \( 2+2\varepsilon \) and \( 2-2\varepsilon \). Or, put another way,

\[
0 < |x-2| < 2\varepsilon \quad (1.1)
\]

But if \( x \) is between \( 2+2\varepsilon \) and \( 2-2\varepsilon \), then \( \frac{1}{2}x \) is between \( 1+\varepsilon \) and \( 1-\varepsilon \), so

\[
\frac{|x|}{2} - 1 < \varepsilon \quad (1.2)
\]

We got from (1.1) to (1.2) with our intuition. However, comparing the two inequalities it also becomes clear how one can arrive at (1.2) with simple algebra. Dividing the first inequality by 2 gives us the second inequality directly, mathematically demonstrating that (1.2) must follow from (1.1).

So what have we shown? \( \frac{4}{2} \) is just \( f(x) \) for \( x \neq 0 \), so we’ve shown that, given \( \varepsilon \), taking \( \delta = 2\varepsilon \) means that if

\[
0 < |x - c| < \delta
\]

then

\[
|f(x) - l| < \varepsilon
\]

But this is just the definition of the limit! (Definition 2) By showing how to find a \( \delta \) (given any \( \varepsilon \)) that works in the definition (i.e by dividing it by 2), we’ve shown that for any \( \varepsilon \), there exists a \( \delta \) (and we know how to find it!) that makes the definition work.

Thus, for \( f(x) = \begin{cases} \frac{1}{2}x & \text{when } x \neq 2 \\ 2 & \text{when } x = 2 \end{cases} \), we’ve shown that

\[
\lim_{{x \to 2}} f(x) = 1
\]

That’s all there is to proving that a value is a function’s limit. We showed how to find a \( \delta \) that satisfied the conditions of Definition 2 no matter what the \( \varepsilon \) given to us. In the next section, we’ll examine more thoroughly the process of proving that a value is a limit of a function, and develop the notions of a right hand limit and left hand limit.

**Problems**

1. Give an example of a function and a point such that the point is a limit of the function according to our first attempted definition of a limit (page 2) but not according to either Definition 1 or 2.

2. Give an example of a function that has a limit according to our second attempted definition of a limit (page 4) but not according to Definition 1 (or
3. We’ll say one definition is stronger than another if the latter holds whenever it holds. So, a the definition of a square is a stronger than that of a rectangle, and that of a rectangle stronger than that of a quadrilateral. Of course, given two definitions, neither may be stronger than the other; the definition of a circle, for example, is neither stronger nor weaker than the definition of a square. In this case we call the two definitions incomparable. Comparing our first attempted definition of a limit (given on page 2) with Definition 1 (or 2), is one stronger than the other (in which case, which is it) or are they incomparable?

Hint. Are there any functions and points where the the definition on page 2 says the point is a limit of the function where Definition 1 doesn’t? What about in the other direction?)

4. Consider the function \( f(x) = \frac{x(x-1)}{2(x-1)} \) (shown in figure 1.1 on page 2). What is it’s limit at \( x = 0 \)? (According to definitions 1 and 2)? Pick three different values for the \( \varepsilon \) in Definition 2, and in each case, find the corresponding \( \delta \) which satisfies the requirements of the definition. What other values for \( \delta \) will work for the same values for \( \varepsilon \)?

5. Do the same as in Problem 4, but with \( h(x) = x^2 \) (see figure 1.2 on page 3).

6. Based on your results from Problem 4, find a formula for \( \delta \) in terms of \( \varepsilon \) for \( f(x) \) that will always satisfy the conditions of Definition 2, and show that it will, just as we did with the \( f(x) \) shown in Figure 1.7. This will prove your assertion about what the limit of \( f(x) \) is at 0 in that problem.

7. Based on your results from Problem 5, find a formula for \( \delta \) in terms of \( \varepsilon \) for \( h(x) \) that will always satisfy the conditions of Definition 2, and show that it will, just as we did with the \( f(x) \) shown in Figure 1.7. This will prove your assertion about what the limit of \( h(x) \) is at 0 in that problem.

1.2 Doing \( \varepsilon-\delta \) Proofs

1.2.1 “Simple” \( \varepsilon-\delta \) Proofs

In Section 1.1, we proved that for

\[
f(x) = \begin{cases} 
\frac{1}{2}x & \text{when } x \neq 2 \\
2 & \text{when } x = 2
\end{cases}
\]

The limit of \( f(x) \) at 2 is 1. In our limit notation, this is just

\[
\lim_{x \to 2} f(x) = 1
\]
To prove this, we needed to find a \( \delta \) (actually a formula for finding a \( \delta \) for a given \( \varepsilon \)) that would always satisfy the requirements of the definition, no matter what \( \varepsilon \) we chose. Proving that 1 was the limit of \( f(x) \) at 2, then, was just a matter of proving that this \( \delta \) that we came up with did behave like this.

What are these “requirements of the definition” we needed our \( \delta \) to satisfy? What does our \( \delta \) have to be able to do? We need a formula that gives us a \( \delta \) (for any \( \varepsilon \)) such that whenever 

\[
0 < |x - c| < \delta \tag{1.3}
\]

we have 

\[
|f(x) - l| < \varepsilon \tag{1.4}
\]

(see Definition 2 on page 7)

The first thing we had to do to prove the limit, of course, was come up with a formula for \( \delta \) that would work. In Section 1.1, we found \( \delta \)'s that worked for specific values of \( \varepsilon \), and then tried to establish a pattern between them which we could use to come up with our general formula for \( \delta \).

Now, we’re going to try to suggest a way of directly finding a formula for \( \delta \) that will work—without having to test specific values.

Consider the function \( f(x) \) at the point 1, where 

\[
f(x) = \begin{cases} 
2x & \text{when } x \neq 1 \\
1 & \text{when } x = 1 
\end{cases}
\]

(See Figure 1.8). This function is very similar to the other we did the proof for—it’s a line with one point where we wouldn’t expect it. This time, however, we’re going to try to find our \( \delta \) without trial and error—that is, we’ll try to mathematically find a plausible \( \delta \) for us to test in the definition, rather than taking the time to come up with trial values and try to find relationships between them. Of course, before we do any of this, we have to hypothesize what we think the limit will be at 1 (remember, our definition can only check our hypothesis). Looking at Figure 1.8, it looks like \( \lim_{x \to 1} f(x) = 2 \) seems like a reasonable hypothesis.

Remember, according to the definition of the limit, our \( \delta \) has to be chosen so that whenever 

\[
0 < |x - 1| < \delta \tag{1.5}
\]

we have 

\[
|2x - 2| < \varepsilon \tag{1.6}
\]
How do we know this is what we need? By plugging in our values of $c$, $l$ into (1.3) and (1.4), respectively, and by substituting $2x$ for $f(x)$. Remember, the “$0 < |x - 1|...$” part of (1.5) means that we can ignore the value $f(x)$ takes at 1.

Let’s pretend we’ve already found our $\delta$. How can we get from (1.5) to (1.6)? Multiplying (1.5) by 2, we get

$$0 < |2x - 2| < 2\delta$$

How can we make this the same as (1.6)? By replacing $\delta$ with $\frac{\delta}{2}$. In this case, we’ll just drop the “$0 <$” from this inequality, as it is just extra information not needed to satisfy the definition. In other cases (see Problem 6), dropping the “$0 <$” won’t just be allowed, but also necessary.

In any case, this suggests a proof of the limit. We’ve found our $\delta$, now we need only to show that it works. Our complete $\varepsilon$–$\delta$ proof, along with the significance of each step, is shown in Table 1.3

<table>
<thead>
<tr>
<th>$\varepsilon$–$\delta$ proof</th>
<th>Step by Step Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Take $\delta = \frac{\varepsilon}{2}$</td>
<td>To prove the limit, you need to show that for any $\varepsilon$, there exists a $\delta$ such that . . . . The proof begins by giving the formula for that $\delta$.</td>
</tr>
<tr>
<td>$0 &lt;</td>
<td>x - c</td>
</tr>
<tr>
<td>$0 &lt;</td>
<td>x - 1</td>
</tr>
<tr>
<td>$0 &lt;</td>
<td>2x - 2</td>
</tr>
<tr>
<td>$</td>
<td>f(x) - 2</td>
</tr>
</tbody>
</table>

Table 1.3: Proof of $\lim_{x \to 2} f(x) = 1$

Notice that the process of finding the formula for the $\delta$ isn’t part of the proof at all. It doesn’t matter how we find our $\delta$—it only matters that it works. Even if we had picked the formula $\delta = \frac{\varepsilon}{2}$ out of hat, it wouldn’t have affected the validity of our proof so long as we showed that it worked in the definition.

What about proving that something isn’t the limit of a function? Look back at Table 1.1 on page 7. If a value is a limit of the function, than for any $\varepsilon$ we can find a $\delta$ that satisfies the conditions of the definition. If the value isn’t the limit of the function, then, it means that there is at least one $\varepsilon$ for which
1. Limits

we will not be able to find a $\delta$ that satisfies the conditions of the definition. We’ve already proved that 2 is the limit of

$$f(x) = \begin{cases} 
2x & \text{when } x \neq 1 \\
1 & \text{when } x = 1 
\end{cases}$$

at the point 1. Now we’ll show that the point 1 is not the limit.

So, for what value of $\varepsilon$ will we not be able to find a $\delta$ that works? Figure 1.9 suggests that $\varepsilon = \frac{1}{2}$ might work. It certainly looks like there is no region around 1 for which $f(x)$ is always within $1 + \frac{1}{2}$ and $1 - \frac{1}{2}$. But how can we prove, mathematically, that this is the case?

Consider any region $(1 - \delta, 1 + \delta)$ around the point 1. We need to show that for any such region, there are $x$ values in the region such that $f(x)$ is not within $\frac{1}{2}$ of 1. Here is the proof:

Any such region must contain values bigger than 1. But for any $x$ bigger than 1, $f(x)$ is bigger than 2. So any region $(1 - \delta, 1 + \delta)$ contains values for $x$ for which $f(x)$ is not within $\frac{1}{2}$ of 1, and 1 can therefore not be the limit of $f(x)$ at 1.

In other words, we took a specific $\varepsilon$ (we used $\varepsilon = \frac{1}{2}$) and showed that no matter what $\delta$ you tried, you would still have values of $x$ in $(1 - \delta, 1 + \delta)$ for which $f(x)$ would not be within that $\varepsilon$ of your $l$ (1 in this case).

Can a function ever have two different limits at one point? That would mean that the function would get arbitrarily close to both points at the same time, which intuitively doesn’t seem possible. Can we prove that it’s impossible? This suggests our first theorem.

**Theorem 1.** A function $f(x)$ can have at most one limit at a point $c$

**Proof.** To prove this, we’ll do a proof by contradiction; that is, we’ll assume that a function could have two limits at one point, and show that this would mean something that we know isn’t true.

First let’s assume that $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} f(x) = m$, where $m \neq l$. What do we know this means? Looking at Definition 2, we know that this means that for any $\varepsilon_l$ around $l$, there is some $\delta_l$ such that for $x$ within $\delta_l$ of $c$, $f(x)$ is within $\varepsilon_l$ of $l$.

But we also know that for any $\varepsilon_m$, there is some $\delta_m$ such that whenever $x$ is within $\delta_m$ of $c$, $f(x)$ is within $\varepsilon_m$ of $m$. Can both of these be true when $m$ and $l$ are two distinct values?
If \( l \) and \( m \) are two different values, call \( d \) the distance \( |l - m| \) between them. Now choose \( \varepsilon_l = \frac{d}{2} \) and \( \varepsilon_m = \frac{d}{2} \), half the distance between the two values (Remember, if these are both limits of the function, we need to be able to pick whatever \( \varepsilon \)'s we want). If both \( l \) and \( m \) are limits of \( f(x) \) at \( c \) (as we have assumed), then there are regions \((c - \delta_l, c + \delta_l)\) and \((c - \delta_m, c + \delta_m)\) for \( x \) such that \( f(x) \) is within \( \frac{d}{2} \) of \( l \) and \( m \) respectively. Since these are both regions around \( c \), there is some region around \( c \) where they overlap. In this region where they overlap, \( f(x) \) must be within \( \frac{d}{2} \) of both \( l \) and \( m \). But since \( m \) and \( l \) are \( d \) apart from each other, it is impossible for \( f(x) \) to be less than \( \frac{d}{2} \) away from \( l \) and less than \( \frac{d}{2} \) away from \( m \). □

What have we done here? We state a theorem—that a function can have at most one limit at any point \( c \), and then we prove that it’s true. In this case, to prove that it’s true, we showed that if it wasn’t true, then something else we know is impossible would have to be true. We showed that if a function could have more than one limit, than it would have to be less than half as close to each of the limits at that point as they are to each other, which we know is impossible.

This theorem is certainly useful—once we have found a limit for a function at a point, this theorem tells us that that limit is the only limit of the function at that point. But wouldn’t it be nice to to have theorems that actually help us find the limit in the first place? This is the subject of section 1.3.

### 1.2.2 Less Simple \( \varepsilon-\delta \) Proofs

We’ve already used Definition 2 to prove some of our suspicions of what limit certain functions had at given points. But these functions have all been rather simple—just lines with a point removed or in a strange place, in most of our examples. But what about more complicated functions? Recall the function \( h(x) = x^2 \) shown in Figure 1.2 on 3. How do we structure the rigorous \( \varepsilon-\delta \) proof for this function?

Let’s start with the simplest case... proving that \( \lim_{x \to 0} h(x) = 0 \). To do this, let’s see where we get from \( 0 < |x - c| < \delta \)
limits

\[ 0 < |x - c| < \delta \]

We’ll need to find a \( \delta \) that will get us from here to \(|f(x) - l| < \varepsilon\)

\[ 0 < |x| < \delta \]

Since we’re taking the limit at 0, \( c = 0 \)

\[ 0 < |x^2| < \delta^2 \]

Square everything: we’re trying to get to \(|x^2 - 0| < \ldots \)

\[ 0 < |x^2| < \delta^2 \]

Something squared is always positive, so it doesn’t matter whether the square is on the inside or outside of the absolute value.

\[ |h(x) - 0| < \delta^2 \]

Substitute \( x^2 = x^2 - 0 = h(x) - 0 \).

So where do we go from here? In this case, it seems that taking \( \delta = \sqrt{\varepsilon} \) will work, and indeed for the limit of \( h(x) \) at 0, this seems to do the job: For \( \delta = \sqrt{\varepsilon} \), the last line from above gives us

\[ |h(x) - 0| < \varepsilon \]

which seems to prove that 0 is the limit of \( h(x) \) at the \( x = 0 \).

There is a problem with this choice of \( \delta \), however: it requires us to know that the square root of any positive number \( \varepsilon \) exists. This seems trivial—any positive number has a square root! Proving this statement is difficult, however, and the proof will use that \( \lim_{x \to c} x^2 = c^2 \) for all numbers \( c \). But this is the statement we’re trying to prove for \( c = 0 \)! We can’t prove things by assuming things that are only true if we know that what we’re trying to prove is true, but we won’t be able to prove that \( \varepsilon \) has a square root without assuming that the limit of \( h(x) \) is 0 at 0 (among other things).

So can we do this proof without taking the square root of \( \varepsilon \)? What about for \( \delta = \varepsilon \)? Then we have

\[ |h(x) - 0| < \varepsilon^2 \quad (1.7) \]

This doesn’t always give us

\[ |h(x) - 0| < \varepsilon \quad (1.8) \]

(which is what we need), but does it ever give us that? If \( \varepsilon^2 \leq \varepsilon \), \emph{then} Eq. 1.8 would have to be true if Eq. 1.7 was true. This means taking \( \delta = \varepsilon \) works in a \( \varepsilon-\delta \) proof if we know that \( \varepsilon^2 \leq \varepsilon \).

When is \( \varepsilon^2 \leq \varepsilon \)? For \( \varepsilon \leq 1 \), of course. For \( \varepsilon > 1 \), squaring \( \varepsilon \) gives you a number bigger than \( \varepsilon \). Hence, \( 2^2 > 2 \) and \( 10^2 > 10 \), whereas \( (\frac{1}{2})^2 < \frac{1}{2} \) and \( .01^2 < .01 \).

So if we can take \( \delta = \varepsilon \) and have it work out for \( \varepsilon \leq 1 \), what should we take as our \( \delta \) when \( \varepsilon > 1 \)? Remember, the problem was that for \( \delta = \varepsilon, \delta^2 \) is
bigger than \( \varepsilon \) when \( \varepsilon \) is bigger than 1. So for \( \varepsilon > 1 \), we need to take \( \delta \) to be a number smaller than \( \varepsilon \). How about \( \frac{1}{2} \), or 1? These are both less than \( \varepsilon \) when \( \varepsilon > 1 \). So whenever \( \varepsilon \) is less than 1, we know we can take \( \delta = \varepsilon \), but when it’s bigger than 1, we’ll just take \( \delta = 1 \). Now let’s try the proof.

**Proof.** Earlier, we showed that \( |x - 0| < \delta \) implied that

\[
|h(x) - 0| < \delta^2
\]  

(1.9)

Substituting \( \delta = \min(\varepsilon, 1) \), we get

\[
|h(x) - 0| < \varepsilon^2 \quad \text{for } \varepsilon \leq 1
\]

\[
|h(x) - 0| < 1^2 = 1 \quad \text{for } \varepsilon > 1
\]

(1.10)

But the first part of Eq. 1.10 means also that \( |h(x) - 0| < \varepsilon \) for \( \varepsilon \leq 1 \), because, as we’ve already observed, \( \varepsilon^2 \leq \varepsilon \) for \( \varepsilon \leq 1 \). But of course, \( 1 < \varepsilon \) when \( \varepsilon > 1 \), so from the second part of Eq. 1.10 we get that \( |h(x) - 0| < \varepsilon \) for \( \varepsilon > 1 \) as well. Combining these two parts into one, we get

\[
|h(x) - 0| < \varepsilon \quad \text{(always!)}
\]  

(1.11)

\( \square \)

And this completes our proof! We’ve shown that there’s always a way of picking a \( \delta \) for a given \( \varepsilon \) so that \( |x - c| < \delta \) means that \( |h(x) - l| < \varepsilon \).

How about another, slightly more complicated example? What if we wanted to prove that \( \lim_{x \to 2} h(x) = 4 \)? We’ll need to be able to get a \( \delta \) that requires

\[
|x^2 - 4| < \varepsilon
\]  

(1.12)

to be true whenever we have that

\[
|x - 2| < \delta
\]  

(1.13)

In other words, we need to be able to make \( |x^2 - 4| \) small by making \( |x - 2| \) small. We begin by factoring (1.12):

\[
|x - 2| \cdot |x + 2| < \varepsilon
\]  

(1.14)

It might seem obvious that the left side of (1.14) will get smaller when we make the left side of (1.13) smaller, but how can we be sure that it will get smaller than \( \varepsilon \)? What if, for example, \( |x + 2| \) is really big? When we have (1.13), just how big can \( |x - 2| \cdot |x + 2| \) be?
Well, if $\delta$ is bigger than $|x - 2|$, then we can be sure that
\[ |x - 2| \cdot |x + 2| < \delta \cdot |x + 2| \]
But (1.13) also means that $x$ can’t be bigger than $2 + \delta$ (Remember, (1.13) means that $x$ is between $2 - \delta$ and $2 + \delta$). This means, then, that $|x + 2|$ is always less than $4 + \delta$, so we can make that substitution as well:
\[ |x - 2| \cdot |x + 2| < \delta \cdot (4 + \delta) \quad (1.15) \]
So how can we be sure that $\delta \cdot (r + \delta)$ is less than $\varepsilon$? If we know that $(4 + \delta)$ is smaller than some number, like 5, this isn’t very hard: $\delta < \frac{5}{5}$ would work in this case. But how can we make $(4 + \delta)$ less than 5? If we require that $\delta$ is never bigger than 1 (as we did in the proof of $\lim_{x \to 0} x^2 = 0$), then $(4 + \delta)$ will in fact always be less than 5.

This suggests that taking $\delta = \min(1, \frac{5}{5})$ will work. Using this $\delta$, can we complete the proof that $\lim_{x \to 2} x^2 = 4$?

**Proof.** Starting with
\[ |x - 2| < \delta \]
we’ve already shown (leading up to equation (1.15)) that
\[ |x - 2| \cdot |x + 2| < \delta \cdot (4 + \delta) \]
Take $\delta = \min(1, \frac{5}{5})$. Then we have
\[ |h(x) - 4| < \frac{5}{5} \cdot (4 + \frac{5}{5}) \quad \text{for } \varepsilon \leq 5 \]
\[ |h(x) - 4| < 1 \cdot (4 + 1) \quad \text{for } \varepsilon > 5 \quad (1.16) \]
But for $\varepsilon \leq 5$, $4 + \frac{5}{5} \leq 5$, so this is the same as
\[ |h(x) - 4| < \frac{5}{5} \cdot 5 \quad \text{for } \varepsilon \leq 5 \]
\[ |h(x) - 4| < 5 \quad \text{for } \varepsilon > 5 \quad (1.17) \]
simplifying further, and noting that for $\varepsilon > 5$, 5 is less than $\varepsilon$,
\[ |h(x) - 4| < \varepsilon \quad \text{for } \varepsilon \leq 5 \]
\[ |h(x) - 4| < \varepsilon \quad \text{for } \varepsilon > 5 \quad (1.18) \]
\[ |h(x) - 4| < \varepsilon \quad \text{(always)} \quad (1.19) \]
Thus we’ve proven that \( \lim_{x \to 2} x^2 = 4 \).

These “less simple” \( \varepsilon-\delta \) proofs require more work and creativity than the linear examples introduced in 1.2.2, and have thus been treated separately in this section. They require that one can identify a choice of \( \delta \) that will work in a given problem as suggested by manipulation of the expression \( |x - c| < \delta \), rather than being given directly by simple computation. Obviously, these more complicated limit proofs can quickly become tedious. We do not fancy the idea of doing an \( \varepsilon-\delta \) proof of the limit of more complicated functions, such as \( \lim_{x \to 0} f(x) = 2x^{-1} \sin(x) = 2 \), every time we want to know that a limit is what we suspect. To avoid this, we will develop a set of limit theorems in Section 1.3 with which will allow us to deduce the limit of a function at a given point without doing an \( \varepsilon-\delta \) proof for every case.

Problems

1. Prove the following statements
   
   i. \( \lim_{x \to 3} x + 3 = 6 \)
      
      \( \text{Hint.} \) Take \( \delta = \varepsilon \)
   
   ii. \( \lim_{x \to 4} 2x + 5 = 13 \)
    
    \( \text{Hint.} \) Take \( \delta = \frac{\varepsilon}{2} \)
   
   iii. \( \lim_{x \to 3} x - 1 = 2 \)
   
   iv. \( \lim_{x \to 2} 3x - 2 = 4 \)
   
   v. \( \lim_{x \to -1} \frac{2x}{x + \frac{1}{3}} = 0 \)
   
   vi. \( \lim_{x \to 2} \frac{2x(x - 2)}{x - 2} = 4 \)
   
   vii. \( \lim_{x \to 5} 5 = 5 \)

2. Give a function \( f(x) \), and specify a point \( c \), such that taking \( \delta = \frac{\varepsilon}{10} \) would not work in an \( \varepsilon-\delta \) proof of the limit of \( f(x) \) at that point.

3. Give an example of a function whose limit at the point 0 equals the value of the function at the point 0.

4. Give an example of a function (not already given in this text) whose limit at the point 0 does not equal the value of the function at the point 0.

   \( \text{Hint.} \) See Fig. 1.9 on page 14.
5. Prove the following statements
   i. \( \lim_{x \to 3} x^2 = 9 \)
   \( \text{Hint. Take } \delta = \min(1, \frac{\epsilon}{7}) \)
   ii. \( \lim_{x \to 1} 2x^2 = 2 \)
   iii. \( \lim_{x \to 0} x^2 + 1 = 1 \)
   iv. \( \lim_{x \to 0} x^3 = 0 \)
   v. \( \lim_{x \to 2} x^3 = 8 \)

6. Prove that \( \lim k = k \) for any \( c \). Why is the “0 <” part of \( 0 < |x - c| < \delta \) important in this case?

7. Prove the following statements
   i. \( \lim_{x \to 1} \frac{1}{x} = 1 \)
   ii. \( \lim_{x \to 1} \sqrt{x} = 1 \)
   iii. \( \lim_{x \to 1} x^2 + \sqrt{x} = 2 \)

8. Prove the following statements
   i. \( \lim_{x \to a} x = a \)
   ii. \( \lim_{x \to a} 2x = 2a \)

9. Prove the following statements
   i. \( \lim_{x \to a} x^2 = a^2 \)
   ii. \( \lim_{x \to a} \frac{1}{x} = \frac{1}{a} \) for \( x > 0 \)

1.3 Limit Theorems

Consider the function \( f(x) = x \). At every point \( c \), we would expect the limit of \( f(c) \) to be \( c \). So, we would expect that \( \lim_{x \to 2} f(x) = 2 \), \( \lim_{x \to 0} f(x) = 0 \), \( \lim_{x \to -7} f(x) = -7 \), and so on. Expecting this to always be the case is nice, but if we could prove that it was always true, then we could state it as a theorem and use it to do other problems without repeating the proof every time. Let’s give it a shot.
A Limit Theorem. For any number $c$,

$$\lim_{x \to c} x = c$$

Proof. For any $\varepsilon$, take $\delta = \varepsilon$. Then starting with . . .

$$0 < |x - c| < \delta$$

make the $\delta = \varepsilon$ substitution:

$$0 < |x - c| < \varepsilon$$

and substitute $f(x)$ for $x$:

$$|f(x) - c| < \varepsilon$$

What about functions like $g(x) = 2x$ or $h(x) = \frac{1}{2}x$? If we know the limit of a function $f(x)$, can we know the limit of some number times that function? We might expect, for example, that if $\lim_{x \to 3} f(x) = 7$, then $\lim_{x \to 3} (2 \cdot f(x)) = 14$. In other words, we would expect that the limit of some number times a function would just be that number times the limit of the function. This would certainly be a useful theorem. Can we prove it?

A Limit Theorem. For any number $k$, if

$$\lim_{x \to c} f(x) = l$$

then

$$\lim_{x \to c} kf(x) = k \cdot l$$

Proof. How will we prove this? How can we prove what the limit of $kf(x)$ is when we don’t know what $k$ and $f(x)$ are? First, we look carefully at our assumptions. The theorem holds if $\lim_{x \to c} f(x) = l$. What does it mean for this condition to hold? Returning to Definition 2, we know this means that for any $\varepsilon$, we can find a $\delta$ such that when

$$0 < |x - c| < \delta$$

we have

$$|f(x) - l| < \varepsilon$$

Our assumption doesn’t tell us anything about what this $\delta$ looks like—just that it exists. In our proof, we’ll call this $\delta$ given for any $\varepsilon$ as $\delta_\varepsilon$ (to show that it
depends on $\varepsilon$). So, we know that

$$0 < |x - c| < \delta \Rightarrow |f(x) - l| < \varepsilon$$  \hspace{1cm} (1.20)

*Note*. The $\Rightarrow$ means “implies”; or, “when the first thing is true, then so is the second.”

And we need to find some $\delta_k$ such that

$$0 < |x - c| < \delta_k \Rightarrow |k f(x) - k l| < \varepsilon_k$$  \hspace{1cm} (1.21)

Dividing both sides of $|k f(x) - k l| < \varepsilon$ by $k$, we see that we need a $\delta_k$ such that

$$0 < |x - c| < \delta_k \Rightarrow |f(x) - l| < \frac{\varepsilon_k}{k}$$  \hspace{1cm} (1.22)

But we know that we can find a $\delta_k$ that satisfies Eq. 1.20 for any $\varepsilon$. So what about for $\varepsilon = \frac{\varepsilon_k}{k}$? Because in Eq. 1.20 we can choose $\varepsilon = \frac{\varepsilon_k}{k}$, there must be some $\delta_k$ such for which Eq. 1.22 holds. Even without knowing the specific formula for this $\delta_k$, we know that it must exist. This proves our theorem. 

This theorem is quite useful: From it, we can use Theorem 1.3 to deduce the limits of functions like $f(x) = 2x$ or $f(x) = \frac{1}{2}x$. But what about functions like $f(x) = x \cdot x$? Given $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} g(x) = m$, it would be nice to know that $\lim_{x \to c} f(x) \cdot g(x) = l \cdot m$. This would allow us to find the limit of a lot more functions—$f(x) = x^2$ would become simple, as $x^2 = x \cdot x$. In fact, $x^n$ for any $n$ would be easy. Combining this with Theorem 1.3, we would be able to find the limit at a point of any function of the form $f(x) = ax^n$. While this would be a very useful theorem, we’ll an also useful, but slightly simpler case.

**Limit Theorem 1.** For $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$,

$$\lim_{x \to a} f(x) + g(x) = l + m$$

*Proof*. We begin by stating what we know: For any $\varepsilon$, there is some $\delta_f$ such that

$$0 < |x - c| < \delta_f \Rightarrow |f(x) - l| < \varepsilon$$

and there is some $\delta_g$ such that

$$0 < |x - c| < \delta_g \Rightarrow |g(x) - m| < \varepsilon$$

Now, as in previous proofs involving limits of $f$ and $g$, we’ll define $\delta = \min(\delta_f, \delta_g)$. Then we know that for this $\delta$,
\[ 0 < |x - c| < \delta \Rightarrow \begin{cases} |f(x) - l| < \varepsilon \\ |g(x) - m| < \varepsilon \end{cases} \] (1.23)

Now we’ll add the two results in (1.23), giving us

\[ 0 < |x - c| < \delta \Rightarrow |f(x) - l| + |g(x) - m| < 2\varepsilon \] (1.24)

Now we’ll use the **triangle inequality**, which states

\[ |a + b| < |a| + |b| \] (1.25)

Taking \((f(x) - l)\) to be \(a\) and \((g(x) - m)\) to be \(b\), we see that

\[ |f(x) - l + g(x) - m| < |f(x) - l| + |g(x) - m| \]

which means that (1.24) gives us (after some regrouping) that

\[ 0 < |x - c| < \delta \Rightarrow |(f(x) + g(x)) - (l + m)| < 2\varepsilon \] (1.26)

Now it seems that we’re so close! We need to show that there’s always a \(\delta\) such that \(0 < |x - c| < \delta\) implies \(|(f(x) + g(x)) - (l + m)| < \varepsilon\). But we know that we can make the number \(2\varepsilon\) from (1.26) as small as we want, and isn’t this enough? Indeed it is: For any number \(\varepsilon_0\) given to us, we can take the \(\varepsilon\) in (1.26) to be \(\frac{\varepsilon_0}{2}\). We know there is some \(\delta\) (given by \(\min(\delta_f, \delta_g)\)) such that (1.26) will hold for \(ep = \varepsilon_0\) (because there is it will hold for some \(\delta\) for any \(\varepsilon\!\!\), giving us

\[ 0 < |x - c| < \delta \Rightarrow |(f(x) + g(x)) - (l + m)| < \varepsilon_0 \] (1.27)

But this is just what the definition of the limit of the function \((f(x) + g(x))\) would look like for \(\lim_{x \to c}(f(x) + g(x)) = l + m\), so this completes our proof.

Is there something funny about this proof? Our last statement in this case was not based on the number \(\varepsilon\), but instead on a number \(\varepsilon_0\). Does this matter? Of course not: We’ve shown that for any number (whether we call it \(\varepsilon\) or \(\varepsilon_0\) is irrelevant), we can find a \(\delta\) which makes the expression \(|(f(x) - g(x)) - (l + m)|\) smaller than that number. In this case, we had already used the number \(\varepsilon\) in part of our proof, so we used \(\varepsilon_0\) instead to represent this number.

Our next two limit theorems we will note prove here, though you’re encouraged to try them. They each require their own tricks, similar in what they accomplish to the triangle inequality as we’ve used it here.
1. Limits

Limit Theorem 2. For \( \lim_{x \to a} f(x) = l \) and \( \lim_{x \to a} g(x) = m \),
\[
\lim_{x \to a} f(x) \cdot g(x) = l \cdot m
\]

Limit Theorem 3. For \( \lim_{x \to a} f(x) = l, \) and \( l \neq 0 \)
\[
\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{l} \cdot m
\]

These three theorems will allow us to evaluate the limits of many functions.

Example 1.3.1 Evaluate \( \lim_{x \to a} x^2 \)
To find the limit of \( x^2 \) at \( a \), we just break it into many functions that we know
the derivative of at \( a \). So we split it up into \( x^2 = x \cdot x \). But we know that
\( \lim_{x \to a} x = a \), and Limit Theorem 2 allows us to just multiply the individual
limits together to get the limit of the whole thing, giving us \( \lim_{x \to a} x^2 = a \cdot a = a^2 \).

Example 1.3.2 Evaluate \( \lim_{x \to a} 3x^2 \)
To find the limit of \( 3x^2 \) at \( a \), we again just break it into two functions that we
know the derivative of at \( a \). So we split it up into \( 3x^2 = 3 \cdot x^2 \). We know that
\( \lim_{x \to a} 3 = 3 \), and we know that \( \lim_{x \to a} x^2 = a^2 \) from Example 1.3.1. Again,
Limit Theorem 2 allows us to just multiply these individual limits together to
give us \( \lim_{x \to a} 3x^2 = 3 \cdot a^2 = 3a^2 \).

Example 1.3.3 Evaluate \( \lim_{x \to a} 5x^4 \)
Just like in the previous examples, we can split this into a series of products:
\( 5x^4 = 5 \cdot x \cdot x \cdot x \cdot x \), so \( \lim_{x \to a} 5x^4 = 5 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \), so \( \lim_{x \to a} 5x^4 = 5 \cdot 2^4 = 80 \)

Example 1.3.4 Evaluate \( \lim_{x \to a} \frac{1}{x} \) for \( a \neq 0 \)
Which limit theorem can we apply here? We know that the limit of \( x \) at \( a \) is
\( a \), so we’ll apply Limit Theorem 3, which tells us that the limit of \( \frac{1}{x} \) at \( a \) must
be \( \frac{1}{\lim_{x \to a} x} = \frac{1}{a} \) as long as \( a \neq 0 \).

Example 1.3.5 Evaluate \( \lim_{x \to a} \frac{x}{x+1} \) for \( a + 1 \neq 0 \)
To do limits of quotients like this, we first split it into a product of “things on
the top” times the fraction of 1 over the “things on the bottom”. So in this
case, we write that
\[
\lim_{x \to a} \frac{x}{x+1} = \left( \lim_{x \to a} x \right) \cdot \left( \lim_{x \to a} \frac{1}{x+1} \right)
\]
We know that the limit at \( a \) of \( x \) is \( a \). Also, applying Limit Theorem 1, we know that the limit at \( a \) of \( x + 1 \) must be \( a + 1 \). Applying Limit Theorem 3, we conclude then that the limit at \( a \) of \( \frac{1}{x + 1} \) must be \( \frac{1}{a + 1} \), as long as \( a + 1 \neq 0 \). Finally, we plug this back into our first statement and get

\[
\lim_{x \to a} \frac{x}{x + 1} = (a) \cdot \left( \frac{1}{a + 1} \right) = \frac{a}{a + 1}
\]

As long as \( a + 1 \neq 0 \)

Problems

1. Evaluate the following limits (in terms of \( a \)):
   
   i. \( \lim_{x \to a} 13x^2 \)
   
   ii. \( \lim_{x \to a} 11x^5 + 12x^2 \)
   
   iii. \( \lim_{x \to a} \sqrt{3x + x^2} \)
   
   iv. \( \lim_{x \to a} \frac{x^3 + 2x^2 + 3x + 1}{11x^5 + 12x^2 + 3x} \)

2. Evaluate the following limits:
   
   i. \( \lim_{x \to 2} 11x^2 \)
   
   ii. \( \lim_{x \to 3} 11x^3 - 12x^2 \)
   
   iii. \( \lim_{x \to 3} 2\sqrt{3x + x^3} \)
   
   iv. \( \lim_{x \to -1} \frac{2x^2 + 3x^4 + 5x + 2}{9x^7 + 13x^5 + 2x} \)
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