Complexity Limitations on Quantum Computation

Lance Fortnow*

Department of Computer Science
University of Chicago
Chicago, IL 60637

John Rogers†

School of CTI
DePaul University
Chicago, IL 60604

Abstract

We use the powerful tools of counting complexity and generic oracles to help understand the limitations of the complexity of quantum computation. We show several results for the probabilistic quantum class **BQP**.

- **BQP** *is low for* **PP**, *i.e.,* $\mathbf{PP}^{\mathbf{BQP}} = \mathbf{PP}$.
- *There exists a relativized world where* $\mathbf{P} = \mathbf{BQP}$ *and the polynomial-time hierarchy is infinite.*
- *There exists a relativized world where* **BQP** *does not have complete sets.*
- *There exists a relativized world where* $\mathbf{P} = \mathbf{BQP}$ *but* $\mathbf{P} \neq \mathbf{UP} \cap \mathbf{coUP}$ *and one-way functions exist. This gives a relativized answer to an open question of Simon.*

1 Introduction

We have seen a surge in interest in quantum computation over the past few years. This interest comes from new and good theoretical models for quantum Turing machines [BV97] and surprising algorithms for factoring [Sho97] and searching [Gro96].

Exactly how much can we accomplish with quantum computers? We bring two powerful tools from computational complexity theory to bear on this question. First we use counting complexity, in particular the **GapP** functions developed by Fenner, Fortnow and Kurtz [FFK94], to give us new upper bounds on quantum complexity classes. Next we use generic oracles to show that the existence of one-way functions does not necessarily imply that quantum computers are more powerful than deterministic ones.

The power of a quantum Turing machine lies in its ability to have its superpositions "cancel" each other. Fenner, Fortnow and Kurtz [FFK94] developed the notion of **GapP** functions, the closure of the #**P** functions under subtraction. The **GapP** functions have some powerful applications based on a similar cancellation effect. We show, perhaps not too surprisingly, a close relationship between **GapP**-definable counting classes and quantum computing. We can use this relationship to obtain new limitations on the complexity of quantum computing.

*Email: fortnow@cs.uchicago.edu. URL: http://www.cs.uchicago.edu/~fortnow. Supported in part by NSF grant CCR 92-53582. Some of this research occurred while the author was visiting the CWI in Amsterdam.

†Email: rogers@cs.depaul.edu. URL: http://www.depaul.edu/~jrogers.

The usual notion of efficient computation is captured by the bounded probabilistic polynomial-time Turing machine. Such a machine accepts an input x either with probability greater than or equal to $2/3$ or with probability less than or equal to $1/3$. In the first case, we say that x is in the language accepted by M and in the second that it is not. The class of languages accepted by these machines is called **BPP**. Replacing the Turing machine with a quantum Turing machine yields the class **BQP**.

We show that **BQP** is contained in the counting class **AWPP**. Based on previous results about the class **AWPP** [Li93], we can show that **BQP** is low for **PP** and so improve the upper bound given by Adleman, DeMarrais and Huang [ADH97]. We can also use oracle results about **AWPP** to get a relativized world where $\mathbf{P} = \mathbf{BQP}$ but the polynomial-time hierarchy is infinite. We also use these techniques to give a relativized world where **BQP** does not have complete sets.

We know that $\mathbf{BPP} \subseteq \mathbf{BQP}$. An important open question is whether or not the containment is strict. Showing the containment strict would require separating **BPP** and **PSPACE**, a presumably difficult task. One approach is to investigate what kinds of conditions would cause a separation between **BPP** and **BQP**. For example, Simon [Sim97] asked whether the existence of one-way functions is sufficient to cause a separation. A one-way function is a one-to-one, honest, polynomial-time computable function whose inverse is not polynomial-time computable.

Our result shows that there is a relativized world in which this implication fails. Although this does not directly refute the implication, it does demonstrate that, if the implication is true, proving it will require techniques that do not relativize.

2 Definitions

2.1 Preliminaries

As usual, Σ denotes the alphabet $\{0, 1\}$ and Σ^* the set of all finite length strings over Σ. A *language* is a subset of Σ^*. The notation $|x|$ denotes the length of string x. We will sometimes need to compare strings and natural numbers. To do so, we will use the polynomial-time computable isomorphism between the nonzero natural numbers and strings that maps a string x to the natural number whose binary representation is 1 followed by x.

The notation $\langle m, n\rangle$ denotes the Rogers [Rog87] pairing function, that is, a polynomial-time computable function that maps the pair of natural numbers m and n one-to-one and onto the natural numbers. Note that, given $\langle m, n\rangle$, we can extract both m and n in polynomial time. Employing the isomorphism defined above allows us to to apply this pairing function to strings: $\langle x, y\rangle$ denotes a pairing of strings from which we can easily extract x and y.

2.2 Models of computation

Our models of computation are the (classical) Turing machine and the quantum Turing machine. Unless otherwise stated, we assume that all machines are polynomial-time bounded. See Hopcroft and Ullman [HU79] for definitions regarding classical Turing machines.

We allow the machines to have oracle access, which means that they are allowed to make membership queries to a language A, called the oracle, and to receive a correct response in constant time. Such machines have a separate query tape and three extra states: a *query* state, a *yes* state, and a *no* state. When a machine is computing relative to an oracle A, it can query whether a string x is in A by writing x on the query tape and entering the query state. If $x \in A$, the computation's next state is the *yes* state, otherwise it is the *no* state.

A language L is in the complexity class **BPP** if there is a classical machine M such that, for every $x \in L$, at least 2/3 of the paths in the computation $M(x)$ are accepting paths and, for every $x \notin L$, no more than 1/3 of the paths are accepting paths.

2.3 Quantum Computation

We will use a simplified model of quantum computation due to Bernstein and Vazirani [BV97]. While simple, this model captures all of the power of quantum computation. See the paper by Bernstein and Vazirani [BV97] for a discussion of this model and the physics underlying it. For a general introduction to quantum computation see the survey by Berthiaume [Ber97].

Consider the transition function of a Turing machine that maps current state and contents under the tape heads to a new state, new values to write under the tape heads and a direction to move the heads. A deterministic Turing machine's transition function has a single output. A probabilistic Turing machine's transition function maps to a distribution on outputs with nonnegative probabilities that add up to one.

A quantum Turing machine's transition function maps to a *superposition* of the outputs where each output gets an *amplitude* which may be a complex value. In the case of **BQP** as defined below, Adleman, DeMarrais and Huang [ADH97] and Solovay and Yao [SY96] show that we can assume these amplitudes take one of the values in $\{-1, -\frac{4}{5}, -\frac{3}{5}, 0, \frac{3}{5}, \frac{4}{5}, 1\}$. Bennett, Bernstein, Brassard and Vazirani [BBBV97] show that we can assume the quantum Turing machine has a single accepting configuration.

The quantum Turing machines we consider here all run in polynomial time and thus have an exponential number of possible configurations.

Suppose that before a transition each configuration C_i has a real amplitude α_i. Consider the L_2 norm of the amplitudes

$$\sqrt{\sum_i \alpha_i^2}$$

A quantum Turing machine is required to preserve this L_2 norm. This is equivalent to the transition matrix U of the configurations being unitary. For real U, U is unitary if the transpose of U is the inverse of U.

To compute the probability of acceptance consider an initial configuration amplitude vector $\vec{\alpha}$ where $\alpha_0 = 1$ for the initial configuration C_0 and $\alpha_i = 0$ for every other configuration. Let $\vec{\beta} = U^t \cdot \vec{\alpha}$ where t is the running time of the Turing machine. The probability of acceptance is β_i^2 where C_i is the accepting configuration.

We can now define **BQP** similar to the definition of **BPP**.

Definition 2.1 *A language L is in* **BQP** *if there is a quantum Turing machine M such that for all x in Σ^*,*

- *If x is in L then $M(x)$ accepts with probability at least two-thirds.*
- *If x is not in L then $M(x)$ accepts with probability at most one-third.*

Similar to **BPP**, though with nontrivial proofs, we can assume the error is exponentially small and that **BQP** machines can simulate deterministic Turing machines and other **BQP** machines as subroutines (see [BV97]).

The class **EQP** (sometimes called **QP**) has the same definition as **BQP** except that we require zero error. It is analogous to **P** in that every computation path halts in polynomial time. Surprisingly, this class appears to be stronger than deterministic polynomial time (see [BV97]).

2.4 Counting Classes

A function f from strings to the natural numbers is in the counting class **#P** if there is a nondeterministic polynomial-time machine M such that $f(x) = m$ iff the computation $M(x)$ has m accepting paths. In order to understand better the complexity of counting classes like **#P**, Fenner, Fortnow and Kurtz [FFK94] defined the **GapP** functions consisting of the closure under subtraction of the set of **#P** functions (and so a **GapP** function's domain is the integers). Equivalently, **GapP** consists of functions $f(x)$ such that for some nondeterministic polynomial-time Turing machine M, $f(x)$ is the difference between the number of accepting and the number of rejecting paths of $M(x)$.

The power of **GapP** functions lie in their closure properties: **GapP** functions are closed under negation, subtraction, exponential addition and polynomial multiplication.

Theorem 2.2 (Fenner-Fortnow-Kurtz) *Let f be a* **GapP** *function and q a polynomial. Then the following are* **GapP** *functions:*

1. $-f(x)$,
2. $\sum_{|y|\leq q(|x|)} f(\langle x, y\rangle)$, *and*
3. $\prod_{0\leq y\leq q(|x|)} f(\langle x, y\rangle)$.

For the rest of the paper, we will assume that the pairing function is implicitly used whenever we have a function of two or more arguments.

We can also define many interesting counting classes using **GapP** functions. For this paper we consider the following classes.

Definition 2.3 *The class* **PP** *consists of those languages L such that for some* **GapP** *function f and all x in Σ^*,*

- *If x is in L then $f(x) > 0$.*
- *If x is not in L then $f(x) < 0$.*

The class **PP** was first defined by Gill [Gil77] as probabilistic polynomial time with unbounded error. Definition 2.3, first given by Fenner, Fortnow, and Kurtz [FFK94], makes the class considerably easier to work with.

Definition 2.4 *The class* **LWPP** *consists of those languages L such that for some* **GapP** *function f, and some polynomial-time computable positive function g, and for all x in Σ^*:*

- *If x is in L then $f(x) = g(1^{|x|})$.*
- *If x is not in L then $f(x) = 0$.*

Definition 2.5 *The class* **AWPP** *consists of those languages L such that for all polynomials q, there is a* **GapP** *function f and a polynomial-time computable function g such that for all strings x in Σ^* and $m \geq |x|$, $0 < f(x, 1^m) < g(1^m)$ and*

- *If x is in L then $f(x, 1^m) \geq (1 - 2^{-q(m)})g(1^m)$.*
- *If x is not in L then $f(x, 1^m) \leq 2^{-q(m)}g(1^m)$.*

The classes **LWPP** and **AWPP** were first defined by Fenner, Fortnow and Kurtz [FFK94] and Fenner, Fortnow, Kurtz and Li [FFKL93]. Though artificial, these classes have some nice properties that we will use to help classify quantum complexity.

2.5 One-way functions

A language L is in the class **UP** if there is a classical machine M that, for every $x \in L$, has exactly one accepting path and has no accepting paths if $x \notin L$.

A polynomial-time computable function f from strings to strings is *one-way* if it is one-to-one, honest, and not invertible in polynomial time. Being *honest* means that there is a polynomial p such that $p(|f(x)|) > |x|$; in other words, honest functions do not map long input strings to short output strings. Grollmann and Selman [GS88] showed that one-way functions exist if and only if $\mathbf{P} \neq \mathbf{UP}$. Note that these one-way functions may not be suitable for cryptographic purposes which require average-case hardness against nonuniform inverters.

2.6 Generic oracles

In trying to show that there is an oracle relative to which a particular proposition P holds, we often begin by defining an infinite set of *requirements*, which are statements about relativized computations. An oracle X *satisfies* (or *forces*) a requirement if the statement of the requirement is true when the computations are performed relative to X. We define the requirements so that, if each is satisfied, the proposition P is true. For example, to make $\mathbf{P}^X \neq \mathbf{NP}^X$, we specify an enumeration of all polynomial-time bounded, deterministic oracle Turing machines: $\{M_i\}_{i\in\omega}$. We then define a nondeterministic machine N and an infinite set of requirements $R = \{R_i\}_{i\in\omega}$, where R_i is the statement: "$L(M_i^X) \neq L(N^X)$." If we construct an oracle X satisfying every R_i then $\mathbf{P}^X \neq \mathbf{NP}^X$.

Defining the requirements is often quite straightforward. The difficulties usually arise when trying to construct the oracle. We avoid some of the difficulties by employing *generic* oracles.

A *condition* is a partial function from Σ^* to $\{0,1\}$. A condition σ *extends* another condition τ if, for all $x \in dom(\tau)$, $\sigma(x) = \tau(x)$. An oracle A extends a condition σ if A's characteristic function extends σ. Two conditions σ and τ are *compatible* if, for all $x \in dom(\sigma) \cap dom(\tau)$, $\sigma(x) = \tau(x)$. They *conflict* otherwise. We will always assume that if a condition is defined on any string of some length then it is defined on all strings of that length.

A condition σ *satisfies* a requirement if any oracle extending σ satisfies it.

A set of conditions S is *dense* if, for every condition τ, there is a condition $\sigma \in S$ that extends τ. It is *definable* if the set $\{\overline{\sigma} : \sigma \in S\}$ belongs to Π^1_1 (see [Rog87]).

Restrictions can be set on conditions to achieve a desired separation. In this paper, we impose the restriction that all conditions have finite domains. In section 4, we will employ $\mathbf{UP} \cap \mathbf{coUP}$-*conditions*, which have the following further restrictions: a condition takes on the value 0 for every string not at an *acceptable length* and it takes on the value 1 for exactly one string at each acceptable length. An acceptable length is an integer in the range of the *tower* function, which has the recursive definition: $tower(0) = 2$, $tower(n+1) = 2^{tower(n)}$. That is, $tower(n)$ is a tower of 2's with height $n+1$.

An oracle A *meets* a set of conditions S if there is some σ in S that is extended by A. A *generic* oracle is one that meets every dense definable set of conditions. A $\mathbf{UP} \cap \mathbf{coUP}$-*generic* oracle is one that meets every dense definable set of $\mathbf{UP} \cap \mathbf{coUP}$-conditions. $\mathbf{UP} \cap \mathbf{coUP}$-generics were first developed by Fortnow and Rogers [FR94] to study the relationship between separability and one-way functions. More background about these oracles and a variety of other generic oracles can be found in that earlier paper and in papers by Blum and Impagliazzo [BI87] and Fenner, Fortnow, and Kurtz [FFK94].

3 Counting Complexity

In this section we show a close connection between counting complexity and quantum computing.

Theorem 3.1

$$\mathbf{BQP} \subseteq \mathbf{AWPP}$$

Theorem 3.1 follows from the following lemma.

Lemma 3.2 *For any quantum Turing machine M running in time bounded by a polynomial $t(n)$, there is a* **GapP** *function f such that for all inputs x,*

$$\Pr(M(x) \text{ accepts}) = \frac{f(x)}{5^{2t(|x|)}}.$$

Proof of Theorem 3.1. Fix a language L in **BQP** and a polynomial q. Let M be a polynomial-time quantum Turing machine that on input $(x, 1^m)$ accepts for x in L with probability at least $1 - 2^{-q(m)}$ and accepts for x not in L with probability at most $2^{-q(m)}$.

Fix x and m with $m \geq |x|$. Then there is a polynomial $t(m)$ that bounds the running time of $M(x, 1^m)$. By Lemma 3.2 there is a **GapP** function f such that $f(x, 1^m)/5^{2t(m)}$ is the acceptance probability of $M(x, 1^m)$. We can thus fulfill the requirements of Definition 2.5 by letting $g(1^m) = 5^{2t(m)}$. ∎

Proof of Lemma 3.2. We can assume that M has at most 2^t configurations. Let U be the transition matrix of M. By the discussion in Section 2.3 we can assume the entries of U are of the form $w/5$ for w an integer between -5 and 5. By the nature of a transition matrix, we can compute the (i, j) entry of U in time polynomial in $|x|$.

Let $V = 5U$ so V has only integral entries. Let $\vec{\alpha}$ be the initial configuration amplitude vector as described in Section 2.3. Let $\vec{\beta} = V^t \cdot \vec{\alpha}$. Note that each β_i, a component of $\vec{\beta}$ corresponding to configuration C_i, is an exponential sum of a polynomial product of polynomial-time computable entries of V. By Theorem 2.2, we have that each β_i is a **GapP** function.

Let $f(x)$ be β_i^2 where C_i is the accepting configuration of $M(x)$. Again by Theorem 2.2 we have $f(x)$ is a **GapP** function. We have that $f(x, m)/5^{t(|x|)^2}$ is the acceptance probability of $M(x)$. ∎

We can now use properties of **AWPP** to better understand the complexity of **BQP**. Lide Li [Li93] gave an upper bound on the complexity of **AWPP**.

Theorem 3.3 (Li) **AWPP** *is low for* **PP**, *i.e.,* $\mathbf{PP}^{\mathbf{AWPP}} = \mathbf{PP}$.

For completeness we sketch the proof of Theorem 3.3.

Proof Sketch. Suppose L is in $\mathbf{PP}^A$ for some A in **AWPP**. By Definition 2.3, there is some $h \in \mathbf{GapP}^A$ such that for $x \in L$, $h(x) \geq 1$ and $h(x) \leq -1$ otherwise. Let M^A be a relativized nondeterministic polynomial-time Turing machine such that $h(x)$ is the difference of the number of accepting and rejecting computations of $M^A(x)$. We assume without loss of generality that for every A and x each computation path of $M^A(x)$ makes the same number of queries.

Pick a polynomial $q(n)$ such that for strings of length n, M^A has less than $2^{q(n)/2}$ computation paths. Let f and g be **GapP** and polynomial-time computable functions defined for A and q as in Definition 2.5. Let N be a nondeterministic polynomial-time Turing machine such that $f(x, 1^m)$ is the difference of the number of accepting and rejecting paths of $N(x, 1^m)$.

Create a new nondeterministic polynomial-time Turing machine M' as follows. On input x, simulate $M^A(x)$. Every time M makes a query y to A, simulate $N(y, 1^{|x|})$. If N accepts then continue the computation of M assuming y is in A. If N rejects then continue the computation of M assuming y is not in A.

By the choice of q, the mistakes made by the wrong simulation, even totaled over every computation path of $M^A(x)$, are not enough to affect the sign of the difference of the number of accepting and rejecting paths of M'. ∎

From Theorem 3.3 we get the same result for **BQP**.

Corollary 3.4 **BQP** *is low for* **PP**.

This improves and simplifies the bound given by Adleman, DeMarrais and Huang [ADH97].

Corollary 3.5 (Adleman-DeMarrais-Huang)

$$\mathbf{BQP} \subseteq \mathbf{PP} \subseteq \mathbf{P}^{\#\mathbf{P}} \subseteq \mathbf{PSPACE}$$

We also have a class containing **BQP** that is not known to contain **NP** as Beigel [Bei94] has a relativized world where **NP** is not low for **PP**.

Fenner, Fortnow, Kurtz and Li [FFKL93] give an interesting collapse for **AWPP** relative to generic oracles. Their proof builds on a connection between decision tree complexity and low-degree polynomials developed by Nisan and Szegedy [NS94].

Theorem 3.6 (FFKL,NS) *If* $\mathbf{P} = \mathbf{PSPACE}$ *(unrelativized) then* $\mathbf{P}^G = \mathbf{AWPP}^G$ *for any generic* G.

We can create an oracle H by starting with an oracle making $\mathbf{P} = \mathbf{PSPACE}$ and joining a generic G to that. Because the polynomial-time hierarchy is infinite relative to generic oracles and because Theorem 3.1 relativizes, we can get some interesting relativized worlds.

Corollary 3.7 *There exists a relativized world where* $\mathbf{P} = \mathbf{BQP}$ *and the polynomial-time hierarchy is infinite.*

This greatly strengthens the result of Bennett, Bernstein, Brassard and Vazirani [BBBV97] giving a relativized world where **NP** is not in **BQP**.

Using a proof similar to that of Theorem 3.1 we get a stronger upper bound for **EQP**.

Theorem 3.8

$$\mathbf{EQP} \subseteq \mathbf{LWPP}$$

Whether Graph Isomorphism can be solved quickly by quantum computers remains an interesting open question. This possibility is consistent with Theorem 3.8 as Köbler, Schöning and Torán [KST92] have show that Graph Isomorphism sits in **LWPP**.

3.1 Extensions

The techniques in our paper can also be used to show bounds on the decision tree complexity of quantum computers. Here we consider the situation where we wish to compute a function $f : \{0,1\}^N \to \{0,1\}$ where access to input bits are only via oracle questions. We typically do not care about running time in this model, only the maximum number of queries on any computation path.

Grover [Gro96] shows how to get a nontrivial advantage with quantum computers: He shows that computing the OR function needs only $O(\sqrt{N})$ queries although deterministically all N input bits are needed in the worst case.

Bernstein and Vazirani [BV97] give a superpolynomial gap and Simon [Sim97] gives an exponential gap. However, both of these gaps require that there are particular subsets of the inputs to which f is restricted.

Beals, Buhrman, Cleve, Mosca and de Wolf [BBC$^+$98] notice that a limitation on the decision tree complexity of quantum computation follows from the techniques of the proof of Theorem 3.6.

Corollary 3.9 *If there is a quantum algorithm computing a function f defined on all of $\{0,1\}^N$ and using t queries then there exists a deterministic algorithm computing f using $O(t^8)$ queries.*

Using other techniques, Beals, Buhrman, Cleve, Mosca and de Wolf [BBC$^+$98] improve Corollary 3.9 to $O(t^6)$ queries and show better bounds for specific functions.

Vereshchagin [Ver94] gives the following useful lemma for proving a relativized lack of complete sets for some classes.

Lemma 3.10 (Vereshchagin) *Under some weak restrictions on complexity classes $\mathcal{A}$ and $\mathcal{B}$, if*

- *the class of boolean functions computed by polylogarithmic depth decision trees of type $\mathcal{A}$ coincides with the class of functions computed by deterministic polylogarithmic depth decision trees and*
- *there exists a promise problem solved by polylogarithmic depth decision trees of of type $\mathcal{B}$ but not by deterministic polylogarithmic depth decision trees*

then there is an oracle where $\mathcal{A}$ does not have sets polynomial-time Turing hard for $\mathcal{B}$.

The following result then follows from Corollary 3.9 and Lemma 3.10.

Corollary 3.11 *There exists a relativized world where* **BQP** *has no hards sets for* **BPP**. *In particular,* **BQP** *has no complete sets in this world.*

Lemma 3.2 shows how to compute the probability acceptance of a quantum Turing machine with a **GapP** function. Fenner, Green, Homer and Pruim [FGHP98] give a result in the other direction.

Theorem 3.12 (FGHP) *For any* **GapP** *function f there exists a polynomial-time quantum Turing machine M and a polynomial p such that for all x,*

$$\Pr(M(x) \text{ accepts}) = \frac{f(x)}{2^{p(|x|)}}.$$

Theorem 3.12 creates a quantum machine with amplitudes contained in $\{0, -1, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\}$. Fenner, Green, Homer and Pruim [FGHP98] note that our Lemma 3.2 holds where amplitudes may be any positive or negative square roots of rational numbers (the value "5" in the statement of Lemma 3.2 may have to be replaced with a different positive integer).

¿From Lemma 3.2 and Theorem 3.12 we immediately get a new characterization of the class $\mathbf{C_{=}P}$. The class $\mathbf{C_{=}P}$ consists of the languages L for which there exists a **GapP** function f such that x is in L exactly when $f(x) = 0$.

Corollary 3.13 *A language L is in $\mathbf{C_{=}P}$ if and only if there exists a polynomial-time quantum Turing machine M such that x is in L exactly when the probability that $M(x)$ accepts is zero.*

Watrous [Wat98] proves similar results for space-bounded quantum Turing machines.

4 One-Way Functions

We show that one-way functions are not sufficient to guarantee the hardness of **BQP**.

Theorem 4.1 *There is an oracle C relative to which one-way functions exist and $\mathbf{P}^C = \mathbf{BQP}^C$.*

Thus, to demonstrate that the existence of one-way functions implies a separation between **BPP** and **BQP** will require nonrelativizing techniques.

We actually prove a stronger result from which Theorem 4.1 follows.

Theorem 4.2 *There is an oracle C relative to which $\mathbf{P}^C = \mathbf{BPP}^C = \mathbf{BQP}^C \neq \mathbf{UP}^C \cap \mathbf{coUP}^C$.*

To prove Theorem 4.2, we need the following theorem due to Bennett, Bernstein, Brassard and Vazirani [BBBV97].

Theorem 4.3 (BBBV) *Let M be an oracle **BQP** machine that runs in time $p(n)$ and let O be an oracle and x an n-bit input. There is a set S such that for all oracles O', if O' differs from O only on a single string and that string is not in S then $|P[M^{O'} \text{ accepts } x] - P[M^O \text{ accepts } x]| \leq \epsilon$, where $|S| \leq 4p^2(n)/\epsilon^2$.*

This theorem states that for an oracle **BQP** Turing machine M and an input x whose length is n, there is a polynomial (in n) sized set S such that, if a string y is not in S, we can change the oracle's answer on y and the probability that M accepts x is still bounded away from 1/2.

Proof of Theorem 4.2. Let H be an oracle relative to which $\mathbf{P} = \mathbf{PSPACE}$ (H can be any **PSPACE**-complete language). Let G be a $\mathbf{UP} \cap \mathbf{coUP}$-generic oracle, which must have exactly one string at lengths that are exponentially far apart. Let $C = H \oplus G = \{0x | x \in H\} \cup \{1y | y \in G\}$. The oracle C represents a relativization that identifies **P** and **PSPACE** (and so $\mathbf{P} = \mathbf{BPP} = \mathbf{BQP}$) and a *re-relativization* that, we will show, separates **P** and $\mathbf{UP} \cap \mathbf{coUP}$ but that still leaves $\mathbf{P} = \mathbf{BPP} = \mathbf{BQP}$.

First we show that $\mathbf{P}^C \neq \mathbf{UP}^C \cap \mathbf{coUP}^C$. Let $L^X = \{0^n | (\exists x) |x| = n - 1 \ \& \ x0 \in X\}$. It's easy to see that $L^G \in \mathbf{UP}^G \cap \mathbf{coUP}^G$ and so is in $\mathbf{UP}^C \cap \mathbf{coUP}^C$. A simple diagonalization argument demonstrates that $L^G \notin \mathbf{P}^G$. Because G is generic with respect to H, $L^C \notin \mathbf{P}^C$.

Next we show that $\mathbf{P}^C = \mathbf{BQP}^C$. Let M be a $\mathbf{BQP}^C$ machine that runs in time $p(n)$. Since G is generic we can assume that M is *categorically* a **BQP** machine, i.e., for any oracle A and input x, $M^A(x)$ accepts with probability greater than or equal to 2/3 or less than or equal to 1/3 (see [BI87]).

Let x be an input of length n. We need to show that there is a deterministic polynomial-time machine N that, relative to C, determines for an input x whether $M^C(x)$ accepts. Because M runs in polynomial time, there are a polynomial number of lengths for strings that M can query in an oracle. Because G has exactly one string at every acceptable length, there are polynomially many strings in G that could affect M's computation on x. Because the strings in G are exponentially far apart, all but at most one are at lengths that are so short that N on input x can query G on every string at those lengths and so find all of them.

So the only string that N needs to worry about is one at a length ℓ that is so large that N would have to query exponentially many strings to be certain of finding it. Call this string y. Even though N cannot find y by searching, it *can* use its access to H to figure out what M would do on input x under the assumption that there are no strings of length ℓ in G.

Let us say that, under this assumption, $M(x)$ accepts. However, we know there is a string of length ℓ in G that could cause M to change its computation and reject x. But Theorem 4.3 says that there is a set of strings S whose cardinality is bounded by $4p^2(n)/\epsilon^2$ such that, if y is not in S, the probability that M changes its computation is less than or equal to ϵ.

Set ϵ to a value strictly less than $1/6$ and say that we know that y (if it exists) is not in S. By Theorem 4.3, the probability that M accepts x is still strictly greater than $1/2$. In other words, x is still in the language accepted M relative to G. So if N knows that y is not in S, it can simply run the simulation of $M(x)$ under the assumption that there is no string of length ℓ in G and output the correct answer.

So how does N determine whether y is in S? It asks for an explicit enumeration of S. That is, it asks the question: "What is the set S of strings of length ℓ such that, if one of those strings is in the oracle, M rejects x?" S has size at most $4c^2p^2(n)$, where $c > 6$. This question can be answered in **PSPACE** without querying G. N can use its access to H to find S in polynomial time. It then queries G for each of those strings. If none of those strings are in G then N accepts input x because that is what M would do. If N finds one of those strings in G, it would then be able to simulate the computation of $M(x)$ with full knowledge of all of the strings in G that could possibly affect that computation. ∎

4.1 Cryptographic One-Way Functions

The assumption $\mathbf{P} \neq \mathbf{UP}$ does not necessarily imply the existence of *cryptographic* one-way functions, i.e., functions not invertible on a large fraction of inputs with nonuniform polynomial-size circuits. Whether there exists a relativized world where $\mathbf{BPP} = \mathbf{BQP}$ and cryptographic one-way functions exist remains an interesting open question.

One possible approach would look at whether $\mathbf{P} = \mathbf{BQP}$ relative to a random oracle since relative to a random oracle cryptographic one-way functions exist (see [IR89]). Showing this would imply that factoring is in $\mathbf{BQP} = \mathbf{BPP}$ and thus factoring is efficiently computable on probabilistic machines [Sho97].

Theorem 4.4 *If* $\mathbf{P} = \mathbf{BQP}$ *relative to random oracle then* $\mathbf{BQP} = \mathbf{BPP}$ *(unrelativized).*

Proof. Let L be in **BQP**, then for every oracle R, L is in $\mathbf{BQP}^R$. Thus by assumption L is in $\mathbf{P}^R$ for most oracles R. Bennett and Gill [BG81] show that every language with this property sits in **BPP**. ∎

However, we could possibly prove **P** = **BQP** for random oracles under some assumption like **P** = **PSPACE**. If this were true with a relativizable proof, we could start with an oracle relativize to which **P** = **PSPACE** and join a random oracle to it. This would yield a relativized world where **P** = **BQP** and cryptographic one-way functions exist.

5 Conclusions

We give results in this paper indicating severe restrictions on the complexity of quantum computing. We conjecture that **BQP** actually contains *no* interesting complexity classes outside of **BPP**.

Still we believe that quantum computing remains a potentially powerful model of computation. Quantum computers can quickly solve some problems not known complete such as factoring [Sho97] and the potential to solve problems such as graph isomorphism and finding a short vector in a lattice. Also quantum computing can give a large increase in speed, for example a quadratic improvement in **NP**-like search problems [Gro96].

Acknowledgments

We would like to thank André Berthiaume, Harry Buhrman, Richard Cleve, Ronald de Wolf, Wim van Dam and John Watrous for a number of illuminating conversations on quantum computation. Also, thanks to Ronald de Wolf for the corrected statement of Theorem 4.3.

References

[ADH97] L. Adleman, J. DeMarrais, and M. Huang. Quantum computability. *SIAM Journal on Computing*, 26(5):1524–1540, 1997.

[BBBV97] C. Bennett, E. Bernstein, G. Brassard, and U. Vazirani. Strengths and weaknesses of quantum computing. *SIAM Journal on Computing*, 26(5):1510–1523, 1997.

[BBC+98] R. Beals, H. Buhrman, R. Cleve, M. Mosca, and R. de Wolf. Quantum lower bounds by polynomials. In *Proceedings of the 39th IEEE Symposium on Foundations of Computer Science*, New York, 1998. IEEE. To appear.

[Bei94] R. Beigel. Perceptrons, PP and the polynomial hierarchy. *Computational Complexity*, 4:314–324, 1994.

[Ber97] A. Berthiaume. Quantum computation. In A. Selman and L. Hemaspaandra, editors, *Complexity Theory Retrospective II*, pages 23–51. Springer, 1997.

[BG81] C. Bennett and J. Gill. Relative to a random oracle, $P^A \neq NP^A \neq co - NP^A$ with probability one. *SIAM Journal on Computing*, 10:96–113, 1981.

[BI87] M. Blum and R. Impagliazzo. Generic oracles and oracle classes. In *Proceedings of the 28th IEEE Symposium on Foundations of Computer Science*, pages 118–126. IEEE, New York, 1987.

[BV97] E. Bernstein and U. Vazirani. Quantum complexity theory. *SIAM Journal on Computing*, 26(5):1411–1473, 1997.

[FFK94] S. Fenner, L. Fortnow, and S. Kurtz. Gap-definable counting classes. *Journal of Computer and System Sciences*, 48(1):116–148, 1994.

[FFKL93] S. Fenner, L. Fortnow, S. Kurtz, and L. Li. An oracle builder's toolkit. In *Proceedings of the 8th IEEE Structure in Complexity Theory Conference*, pages 120–131. IEEE, New York, 1993.

[FGHP98] S. Fenner, F. Green, S. Homer, and R. Pruim. Determining acceptance possibility for a quantum computation is hard for PH. Technical Report 98-008, Computer Science Department, Boston University, 1998.

[FR94] L. Fortnow and J. Rogers. Separability and one-way functions. In *Proceedings of the 5th Annual International Symposium on Algorithms and Computation*, volume 834 of *Lecture Notes in Computer Science*, pages 396–404. Springer, Berlin, 1994.

[Gil77] J. Gill. Computational complexity of probabilistic complexity classes. *SIAM Journal on Computing*, 6:675–695, 1977.

[Gro96] L. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the 28th ACM Symposium on the Theory of Computing*, pages 212–219. ACM, New York, 1996.

[GS88] J. Grollmann and A Selman. Complexity measures for public-key cryptosystems. *SIAM Journal on Computing*, 17:309–355, 1988.

[HU79] J. Hopcroft and J. Ullman. *Introduction to Automata Theory, Languages and Computation.* Addison-Wesley, Reading, Mass., 1979.

[IR89] R. Impagliazzo and S. Rudich. Limits on the provable consequences of one-way permutations. In *Proceedings of the 21st ACM Symposium on the Theory of Computing*, pages 44–61. ACM, New York, 1989.

[KST92] J. Köbler, U. Schöning, and J. Torán. Graph isomorphism is low for PP. *Computational Complexity*, 2(4):301–330, 1992.

[Li93] L. Li. *On the counting functions.* PhD thesis, University of Chicago, 1993. Department of Computer Science TR 93-12.

[NS94] N. Nisan and M. Szegedy. On the degree of boolean functions as real polynomials. *Computational Complexity*, 4(4):301–313, 1994.

[Rog87] H. Rogers. *Theory of Recursive Functions and Effective Computability.* MIT Press, Cambridge, Massachusetts, 1987.

[Sho97] P. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 26(5):1484–1509, 1997.

[Sim97] D. Simon. On the power of quantum computation. *SIAM Journal on Computing*, 26(5):1474–1483, 1997.

[SY96] R. Solovay and A. Yao, 1996. Manuscript.

[Ver94] N. Vereshchagin. Relativizable and non-relativizable theorems in the polynomial theory of algorithms. *Russian Academy of Sciences Izvestiya Mathematics*, 42(2):261–298, 1994.

[Wat98] J. Watrous. Relationships between quantum and classical space-bounded complexity classes. In *Proceedings of the 13th IEEE Conference on Computational Complexity*, pages 210–227. IEEE, New York, 1998.