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Sensitivity, block sensitivity, and ℓ-block sensitivity of boolean functions

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Abstract

Sensitivity is one of the simplest, and block sensitivity one of the most useful, invariants of a boolean function. Nisan [SIAM J. Comput. 20 (6) (1991) 999] and Nisan and Szegedy [Comput. Complexity 4 (4) (1994) 301] have shown that block sensitivity is polynomially related to a number of measures of boolean function complexity. The main open question is whether or not a polynomial relationship exists between sensitivity and block sensitivity. We define the intermediate notion of ℓ-block sensitivity, and show that, for any fixed ℓ, this new quantity is polynomially related to sensitivity. We then achieve an improved (though still exponential) upper bound on block sensitivity in terms of sensitivity. As a corollary, we also prove that sensitivity and block sensitivity are polynomially related when the block sensitivity is $\Omega(n)$.

1. Introduction

There are several useful measures of the complexity of a boolean function which can be stated without any particular model of computation in mind. Two such measures are sensitivity and block sensitivity. Whether or not these measures are polynomially related is a major open question. We introduce a related measure, ℓ-block sensitivity, which we use to prove that sensitivity and block sensitivity are polynomially related in some special cases, and to narrow the previously known gap in the general case.

The sensitivity of a boolean function at a particular input is the number of input positions where changing that one bit changes the output. The sensitivity of the function is the maximum sensitivity

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of any input. Turán [13] proves a lower bound on the sensitivity of graph properties. Cook et al. [4] use sensitivity to derive a lower bound on parallel time for Concurrent Read Exclusive Write (CREW) parallel RAMs. Deciding whether this bound is tight is still an open problem.

Nisan [8] has introduced the related notion of block sensitivity: instead of counting input positions, we count disjoint subsets of input positions where changing all bits in the subset changes the output. Nisan proves tight bounds for parallel RAM computation in terms of block sensitivity. Block sensitivity has been shown to be polynomially related to other measures of complexity, such as decision tree depth [8], polynomial degree [9], and quantum oracle complexity [1].

Others [2,3,14] have discussed other notions of sensitivity. In particular, Bernasconi [2] defines average block sensitivity and shows that it can be exponentially larger than average sensitivity.

The natural open question is whether sensitivity and block sensitivity are polynomially related. Gotsman and Linial [5] prove this question equivalent to a seemingly-unrelated problem in graph theory. Rubinstein [10] exhibits a family of functions where the gap is quadratic, and this is the largest known separation. However, the best known upper bound on block sensitivity is exponential in sensitivity—this upper bound is a consequence of a result of Simon [12] relating sensitivity to the number of input variables.

We introduce a variant of block sensitivity: ℓ-block sensitivity. This is the same as block sensitivity, except we only allow blocks of size at most ℓ. Our main result is Theorem 3.1: for any ℓ, there is at most a degree-ℓ gap between sensitivity and ℓ-block sensitivity.

We then show two consequences of this main result: Theorem 4.1 states that, if block sensitivity is $\Omega(n)$, where n is the number of input variables, then block sensitivity and sensitivity are polynomially related. Theorem 4.2 gives an improved (though still exponential) upper bound on block sensitivity in terms of sensitivity.

In Section 2, we provide explicit definitions of the different versions of sensitivity we consider, and state the previously known bounds. In Section 3, we prove our main result relating sensitivity and ℓ-block sensitivity. In Section 4, we prove the two consequences mentioned above. In Section 5, we prove some lemmas indicating that the technique used by Rubinstein to construct a quadratic gap cannot be used to construct a larger gap between sensitivity and block sensitivity. Finally, in Section 6, we list some open questions.

Note 1.1. A preliminary version of this paper appeared as a technical report [7]. The earlier version included Definition 2.3 and Theorems 3.1 and 4.1. The proof of Theorem 3.1 in the present paper is more elegant, and yields an improved constant necessary for Theorem 4.2. Eq. (2) was also proved independently by Vishwanathan [15].

2. Preliminaries

2.1. Definitions

Throughout, we let f be a boolean function, $f\colon\{0,1\}^n \to \{0,1\}$. For any subset S of $[n]$, and $x \in \{0,1\}^n$, we form x^S by complementing those bits in x indexed by elements of S. We sometimes write x^i for $x^{\{i\}}$.

Definition 2.1. The *sensitivity* of f at an input x, $s(f;x)$, is the number of indices i such that $f(x) \neq f(x^i)$.

The *sensitivity* of f, denoted $s(f)$, is $\max_x s(f;x)$. For $z \in \{0,1\}$, we also write $s^z(f)$ for $\max_{f(x)=z} s(f;x)$.

For $I \subset [n]$, we say that f is *sensitive on I* if, for every $i \in I$, $f(x) \neq f(x^i)$.

The following definition is due to Nisan [8].

Definition 2.2. The *block sensitivity* of f at an input x, $bs(f;x)$, is the maximum number of disjoint subsets $B_1, \ldots, B_r$ of $[n]$ such that, for all j, $f(x) \neq f(x^{B_j})$.

We refer to such a set B_j as a *block*.

The *block sensitivity* of f, denoted $bs(f)$, is $\max_x bs(f;x)$. For $z \in \{0,1\}$, we also write $bs^z(f)$ for $\max_{f(x)=z} bs(f;x)$.

By considering the partition where every B_i is a singleton, we see that block sensitivity is at least as large as sensitivity. For monotone functions (i.e., functions such that if $f(x)=0$ and the ith coordinate of x is 1, then $f(x^i)=0$), it is easy to check that the two quantities are the same. As stated in Section 1, the major open question is: is there a polynomial relationship between $s(f)$ and $bs(f)$? In other words, do there exist K, d such that

$$bs(f) \leqslant K(s(f))^d$$

for all functions f?

We now introduce another variation on sensitivity:

Definition 2.3. The *ℓ-block sensitivity* of f at an input x, $bs_\ell(f;x)$, is the maximum number of disjoint subsets $B_1, \ldots, B_r$ of $[n]$, such that, for all j, $|B_j| \leqslant \ell$ and $f(x) \neq f(x^{B_j})$.

The *ℓ-block sensitivity* of f, denoted $bs_\ell(f)$, is $\max_x bs_\ell(f;x)$. For $z \in \{0,1\}$, we also write $bs_\ell^z(f)$ for $\max_{f(x)=z} bs_\ell(f;x)$.

Remark. It is clear that $s(f) = bs_1(f)$. Also, if B is a minimal set such that $f(x^B) \neq f(x)$, then $f((x^B)^i) \neq f(x^B)$ for all $i \in B$. We can conclude that $|B| \leqslant s(f)$, and hence that

$$bs(f) = bs_{s(f)}(f).$$

2.2. Previous bounds

The largest known gap between sensitivity and block sensitivity is due to Rubinstein [10]:

Theorem 2.1. *There exists a family of functions f for which*

$$bs(f) = \frac{1}{2}s(f)^2.$$

Proof. For any even m, let $g(x)$ be a boolean function on m variables such that $g(x)=1$ when $x_{2j-1}=x_{2j}=1$ for some j and all other input bits are 0. We note that $s(g)=m$, $s^0(g)=1$, and $bs^0(g)=m/2$.

Now, let $f(x)$ be the function on m^2 variables given by taking an OR of m disjoint copies of g. It follows that $s(f)=m$, and $bs(f)=m^2/2$. □

We will show in Section 5 that this technique cannot be used to construct a superquadratic gap. For the construction above, note that block sensitivity is the same as 2-block sensitivity; we will show in Section 3 that there is at most a quadratic gap between sensitivity and 2-block sensitivity.

Remark. Rubinstein's construction can be generalized as follows: choose some $\ell \geqslant 2$. For any m which is a multiple of ℓ, let $g(x)$ be a boolean function on m variables such that $g(x)=1$ when $x_{j\ell-(\ell-1)}=x_{j\ell-(\ell-2)}=\cdots=x_{j\ell}=1$ for some j and all other input bits are 0. Then, as above, let $f(x)$ be an OR of m disjoint copies of g. It follows that $s(f)=bs_{\ell-1}(f)=m$, and $bs_\ell(f)=m^2/\ell$.

The best previously known upper bound on block sensitivity in terms of sensitivity is due to Simon [12]. He shows that, for any function f depending essentially on n variables (i.e., for any coordinate index i, there is some input x where $f(x)$ differs from $f(x^i)$),

$$s(f) \geqslant \tfrac{1}{2}\log_2 n - \tfrac{1}{2}\log_2\log_2 n + \tfrac{1}{2}.$$

Wegener [16] shows that this is tight up to an additive $O(\log\log n)$.

Since block sensitivity is at most n, and the right-hand side is a monotone increasing function of n, this immediately implies that

$$s(f) \geqslant \tfrac{1}{2}\log_2 bs(f) - \tfrac{1}{2}\log_2\log_2 bs(f) + \tfrac{1}{2}.$$

If we turn this into an upper bound on block sensitivity, we get

$$bs(f)=O\left(s(f)4^{s(f)}\right). \tag{1}$$

3. Sensitivity and ℓ-block sensitivity

We now show that, for any fixed ℓ, ℓ-block sensitivity is polynomially related to sensitivity.

Theorem 3.1. *For* $2 \leqslant \ell \leqslant s(f)$, *and* $z \in \{0,1\}$,

$$bs_\ell^z(f) \leqslant \frac{4}{\ell} s^{1-z}(f) bs_{\ell-1}^z(f). \tag{2}$$

Also,

$$bs_\ell^z(f) \leqslant c_\ell s^z(f)\left(s^{1-z}(f)\right)^{\ell-1}, \tag{3}$$

where

$$c_\ell = \frac{(1+\frac{1}{\ell-1})^{\ell-1}}{(\ell-1)!} < \frac{e}{(\ell-1)!}.$$

Corollary 3.1. $s(f) \geqslant (bs_\ell(f)/c_\ell)^{1/\ell}$, *where* c_ℓ *is the constant of Theorem* 3.1.

The general idea behind the proof is as follows: instead of working directly with $bs_\ell(f;x)$, we compute a weighted sum $t(x)$, in which small blocks count more heavily than large blocks. We require each block of size at most ℓ to have weight at least 1, so an upper bound on t gives an upper bound on $bs_\ell(f)$. We work with the input x maximizing $t(x)$.

We then evaluate $t(x^i)$ for certain choices of i. If i lies in a block of size ℓ, then x^i has a block of size $\ell-1$, which leads to an increase in the sum t. If there is some block A which is no longer a block for x^i, there is a decrease in the sum t. Since x is the point where t is maximized, the decreases must outweight the increase.

We can bound the total decrease, over all choices of i, using sensitivity. If A is not a block for x^i, then x^A is sensitive at i; for each A, there can be at most $s(f)$ such i. The bound we obtain implies a bound on the total increase to t, which in turn gives a bound on the number of blocks of size ℓ.

Proof (*Theorem* 3.1)**.** Fix $z \in \{0,1\}$, and let $s = s^{1-z}(f)$. Let $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_\ell = 1$ be a sequence of weights to be determined later. Given an input x, and a collection $\mathcal{B}$ of disjoint blocks B such that $1 \leqslant |B| \leqslant \ell$ and $f(x^B) \neq f(x)$, define $t(x,\mathcal{B}) = \sum_{B\in\mathcal{B}} w_{|B|}$. We observe that, for any x, $\max_{\mathcal{B}} t(x,\mathcal{B}) \geqslant bs_\ell(f;x)$.

We choose x and $\mathcal{B}$ maximizing $t(x,\mathcal{B})$, subject to $f(x)=z$; for this x and $\mathcal{B}$, we have $t(x,\mathcal{B}) \geqslant bs_\ell^z(f)$. We may assume that each $B \in \mathcal{B}$ is minimal (i.e., x^B is sensitive on B).

For each coordinate index k in some $B \in \mathcal{B}$ with $|B| \geqslant 2$, we define $\mathcal{A}_k$ as follows: $\mathcal{A}_k$ contains the block $B \setminus \{k\}$, and, for each other block $A \in \mathcal{B}$, $A \neq B$, we say $A \in \mathcal{A}_k$ if $f((x^k)^A) \neq f(x^k)$. Note that $\mathcal{A}_k$ is a collection of disjoint blocks of size at most ℓ. We then have that

$$t(x^k,\mathcal{A}_k) = t(x,\mathcal{B}) + (w_{|B|-1} - w_{|B|}) - \sum_{A \text{ s.t. } f((x^k)^A) = f(x^k)} w_{|A|}.$$

For any set $A \in \mathcal{B}$, the term $w_{|A|}$ appears above only if $f((x^A)^k) = f(x^k)$. Since B is minimal and $|B| \geqslant 2$, $f(x^k) = f(x)$. Since $A \in \mathcal{B}$, $f(x) \neq f(x^A)$. Thus $f((x^A)^k) \neq f(x^A)$. Since x^A is already sensitive on A, and $f(x^A) = 1-z$, this can happen for at most $s - |A|$ values of k. So, summing over all k, and letting m_i denote the number of sets in $\mathcal{B}$ of size i, we obtain

$$\sum_k (t(x,\mathcal{B}) - t(x^k,\mathcal{A}_k)) \leqslant \sum_{i=1}^{\ell} m_i w_i (s-i) - \sum_{i=2}^{\ell} i m_i (w_{i-1} - w_i).$$

Since $t(x, \mathcal{B})$ is maximal, the left-hand side must be nonnegative, so

$$\sum_{i=1}^{\ell} m_i w_i (s-i) \geqslant \sum_{i=2}^{\ell} i m_i (w_{i-1} - w_i). \tag{4}$$

Now, to prove Eq. (2), consider what happens when we let $w_i = w > 1$ for $i < \ell$. Then Eq. (4) reduces to

$$\sum_{i=1}^{\ell-1} m_i w (s-i) + m_\ell (s-\ell) \geqslant \ell m_\ell (w-1).$$

This implies that

$$sw \left(\sum_{i=1}^{\ell-1} m_i \right) \geqslant m_\ell (\ell w - s),$$

and therefore, assuming $w > s/\ell$ so that $\ell w - s > 0$, the following easy algebraic manipulation gives:

$$\left(\frac{sw}{\ell w - s} + w \right) \left(\sum_{i=1}^{\ell-1} m_i \right) \geqslant m_\ell + w \left(\sum_{i=1}^{\ell-1} m_i \right) = t(x, \mathcal{B}) \geqslant bs_\ell^z(f).$$

The function $(sw/(\ell w - s)) + w$ is minimized when $w = 2s/\ell$, at which point $(sw/(\ell w - s)) + w = 4s/\ell$. Therefore, we have

$$bs_\ell^z(f) \leqslant \frac{4s}{\ell} \sum_{i=1}^{\ell-1} m_i \leqslant \frac{4}{\ell} s \; bs_{\ell-1}^z(f),$$

where $s = s^{1-z}(f)$. Note that, in particular, $bs_2(f) \leqslant 2s^0(f)s^1(f)$.

Eq. (2) is sufficient to prove a bound of the form $bs_\ell(f) \leqslant Ks(f)^\ell$. However, we can improve the constant K by assigning the weights w_i more carefully; this will give us a proof of Eq. (3). (We will need the improved constant for Theorem 4.2.) First, going back to Eq. (4), we note that

$$\sum_{i=1}^{\ell} m_i w_i (s-i) - \sum_{i=2}^{\ell} i m_i (w_{i-1} - w_i) = m_1 w_1 (s-1) + \sum_{i=2}^{\ell} m_i (s w_i - i w_{i-1}).$$

Now, assume we choose the weights so that, for $2 \leqslant i \leqslant \ell$,

$$i w_{i-1} - s w_i = y w_i, \tag{5}$$

where y is to be determined later. Then we can conclude that

$$m_1 w_1 (s-1+y) \geqslant y m_1 w_1 + \sum_{i=2}^{\ell} y m_i w_i = yt(x, \mathcal{B}) \geqslant y bs_\ell(f).$$

Since m_1 is the number of singletons in $\mathcal{B}$, we know that $m_1 \leqslant s^z(f)$. Solving Eq. (5) for w_{i-1}, we get $w_{i-1} = (s+y)w_i/i$; combining this with $w_\ell = 1$, we have $w_1 = (s+y)^{\ell-1}/\ell!$. We can therefore conclude that

$$\frac{s^z(f)(s+y)^{\ell-1}(s-1+y)}{\ell!} \geqslant y bs^z_\ell(f).$$

It remains to determine what value of y gives us an optimal bound. For simplicity of calculation, we replace $(s-1+y)$ above with $(s+y)$; we know that

$$\frac{s^z(f)(s+y)^{\ell}}{y\ell!} \geqslant bs^z_\ell(f).$$

The function $(s+y)^\ell/y$ is minimized when $y = s/(\ell-1)$. If we use this value of y, we get:

$$\begin{aligned} bs^z_\ell(f) &\leqslant \frac{s^z(f)(s+\frac{s}{\ell-1})^\ell}{s\ell!/(\ell-1)} = \frac{s^z(f)s^{\ell-1}(\ell-1)(1+\frac{1}{\ell-1})^\ell}{\ell!} = \frac{s^z(f)s^{\ell-1}(1+\frac{1}{\ell-1})^{\ell-1}}{(\ell-1)!} \\ &= c_\ell s^z(f)s^{\ell-1}, \end{aligned}$$

which gives Eq. (3). □

Remark. It should be noted that Eq. (2) is tight (up to a constant): for any $\ell \geqslant 2$, the construction in Remark 2.2 yields a function f for which $bs^z_\ell(f) = s^{1-z}(f) bs^z_{\ell-1}(f)/\ell$.

4. Sensitivity and block sensitivity

Theorem 3.1 allows us to prove new relationships between sensitivity and block sensitivity.

Theorem 4.1. *If $bs(f) \geqslant Kn$, then $s(f) = \Omega(n^{K/2})$.*

Proof. Choose an input x which maximizes $bs(f; x)$. The bits of x can be partitioned into $bs(f)$ disjoint blocks $B_1, \ldots, B_{bs(f)}$ with $f(x^{B_i}) \neq f(x)$ for each i. The size of $B_1 \cup \cdots \cup B_{Kn}$ is at most n, and so the average size of B_i in that range is at most $1/K$. Thus by Markov's inequality at least $Kn/2$ blocks have size smaller than $2/K$, and $bs_{2/K}(f) \geqslant Kn/2$.

From Theorem 3.1, we get

$$Kn/2 \leqslant c_{2/K} s(f)^{2/K},$$

and therefore $s(f) = \Omega(n^{K/2})$. □

Eq. (1) gives the best previously known bound on the gap between sensitivity and block sensitivity for all boolean functions. The next theorem reduces that gap.

Theorem 4.2. $bs(f) \leqslant \frac{e}{\sqrt{2\pi}} \mathrm{e}^{s(f)} \sqrt{s(f)}$.

Proof. Let s denote $s(f)$. We note that $bs(f) = bs_s(f)$. Hence, by Theorem 3.1,

$$bs(f) < \frac{e}{(s-1)!} s^s = \frac{e s^{s+1}}{s!}.$$

By Stirling's formula, $s! > (s/e)^s \sqrt{2\pi s}$, so

$$bs(f) < \frac{\mathrm{e}^{s+1} s^{s+1}}{s^s \sqrt{2\pi s}} = \mathrm{e}^{s+1} \sqrt{\frac{s}{2\pi}}. \qquad \square$$

Corollary 4.1. $s(f) \geqslant \ln bs(f) - \frac{1}{2} \ln \ln bs(f) - \ln \frac{e}{\sqrt{2\pi}}$.

5. Block sensitivity and certificate complexity

We now recall the definition of certificate complexity and make several statements relating it to block sensitivity.

Definition 5.1. A *certificate* for f at an input x is a subset S of $[n]$ so that, if $y_i = x_i$ for all $i \in S$, it must be true that $f(y) = f(x)$. The *certificate complexity* of f at x, $C(f; x)$, is the length of the shortest certificate for f at x.

The *certificate complexity* of f, denoted $C(f)$, is $\max_x C(f; x)$. For $z \in \{0, 1\}$, we also write $C^z(f)$ for $\max_{f(x)=z} C(f; x)$.

The next two lemmas are due to Nisan [8], and imply that there is at most a quadratic gap between block sensitivity and certificate complexity.

Lemma 5.1. *For* $z \in \{0, 1\}$, $bs^z(f) \leqslant C^z(f)$.

Proof. For an input x, let $B_1, \ldots, B_r$ be a system of blocks on which x is sensitive. Then any certificate for x must overlap each block B_j, and each certificate must have length at least r. We conclude that $C(f; x) \geqslant bs(f; x)$ for all x, and the result follows. $\square$

Lemma 5.2. *For* $z \in \{0, 1\}$, $C^z(f) \leqslant bs^z(f) s^{1-z}(f)$.

Proof. For an input x, with $f(x) = z$, let $r = bs(f; x)$. Let $B_1, \ldots, B_r$ be a system of disjoint blocks with $f(x^{B_j}) \neq f(x)$. We can assume that each block is minimal (i.e., $f(x^A) = f(x)$ when A is a proper subset of some B_j). Then $f(x^{B_j})$ is sensitive on B_j, so we have $|B_j| \leqslant s^{1-z}(f)$, and hence $|\cup_j B_j| \leqslant bs(f;x)s^{1-z}(f)$. Since $\{B_j\}$ is a maximal system of blocks, $\cup_j B_j$ must be a certificate for x; we conclude that $C(f;x) \leqslant bs(f;x)s^{1-z}(f)$, and the result follows. □

The above lemmas relate $C^z(f)$ and $bs^z(f)$. We now relate $C^{1-z}(f)$ and $bs^z(f)$.

Lemma 5.3. *For $z \in \{0,1\}$, $bs^z(f) \leqslant 2C^{1-z}(f)s^z(f)$.*

Proof. Without loss of generality, we assume that $z=0$, that $f(0)=0$, and that $bs(f;0)=bs^0(f)$. Let $B_1,\ldots,B_r$ be a set of disjoint blocks with $f(0^{B_j})=1$, and $r=bs^0(f)$.

For each k, choose some certificate S_k for 0^{B_k} where $|S_k| \leqslant C^1(f)$. For $j \neq k$, we say that block B_j affects B_k if $S_k \cap B_j$ is nonempty. Consider a directed graph G with r vertices labeled by $\{1,\ldots,r\}$, and with an edge from j to k if B_j affects B_k. Then each vertex in G has indegree at most $C^1(f)$. In fact, the indegree is strictly smaller than $C^1(f)$, since $S_k \cap B_k$ is nonempty.

We claim that G has an independent set H of size $r/(2C^1(f))$. We can construct H using a greedy algorithm: since every vertex has indegree smaller than $C^1(f)$, the average outdegree is also smaller than $C^1(f)$, so there is some vertex v with total degree smaller than $2C^1(f)$. We add v to H, cross off v and its neighbors, and repeat. When the algorithm terminates, H contains at least $r/(2C^1(f))$ vertices.

Now, let $B=\cup_{j\in H}B_j$. Let X be the set of inputs x such that $x_i=0$ for $i \notin B$, and such that $f(x)=0$. Choose y in X maximizing the number of indices i for which $y_i=1$, and let S be the set of indices i such that $i \in B$ but $y_i=0$.

It follows from our choice of y that, for any $i \in S$, $f(y^i)=1$, so we know that $|S| \leqslant s(f;y) \leqslant s^0(f)$. However, if $B_j \cap S=\emptyset$ for any $j \in H$, we would have $y_i=(0^{B_j})_i$ for all i in S_j, and hence $f(y)$ would be 1. So $B_j \cap S \neq \emptyset$ for all $j \in H$, and $|S| \geqslant |H|$. We conclude that

$$s^0(f) \geqslant |S| \geqslant \frac{bs^0(f)}{2C^1(f)}$$

and the result follows. □

These lemmas allow us to prove the following relationship between sensitivity and block sensitivity:

Lemma 5.4. *Let $\phi(\ell)=2c_\ell \ell^{\ell+1}$, where c_ℓ is the constant of Theorem* 3.1. *Then*

$$bs(f) \leqslant \phi(s^0(f))s^1(f).$$

Proof. By combining Lemmas 5.2 and 5.3, we get

$$bs^0(f) \leqslant 2C^1(f)s^0(f) \leqslant 2bs^1(f)(s^0(f))^2.$$

Also, we know that

$$bs^1(f) = bs^1_{s^0(f)}(f).$$

We conclude that

$$bs(f) \leqslant 2bs^1_{s^0(f)}(f)(s^0(f))^2.$$

By Theorem 3.1, we have, for any ℓ,

$$bs^1_\ell(f) \leqslant c_\ell s^1(f)(s^0(f))^{\ell-1}.$$

Taken together, these statements imply that

$$bs(f) \leqslant \phi(s^0(f))s^1(f). \qquad \square$$

Remark. As we mentioned in Section 2.2, Rubinstein [10] exhibits a quadratic gap between sensitivity and block sensitivity. He uses an auxiliary function g on m variables where $s^0(g)=1$ and $bs^0(g)=m/2$. Lemma 5.4 implies that, if $s^0(g)$ is bounded by a constant, then $s^1(g)$ and $bs(g)$ are linearly related. This proves that there is no simple modification of Rubinstein's argument which yields a superquadratic gap between sensitivity and block sensitivity.

6. Open questions

We are still left with the main open question with which we started: what is the relationship between sensitivity and block sensitivity? There are also other questions which come to mind.

1. Do there exist constants K, d such that

   $$s(f) \geqslant K(bs(f))^{1/d}?$$

2. For most "nice" functions, it appears that $s^0(f)s^1(f)=\Omega(n)$. Does this hold for all functions invariant under a permutation group acting transitively on the indices? This would clearly imply that the separation between $s(f)$ and $bs(f)$ (or, indeed, between $s(f)$ and many other complexity measures) is at most quadratic for such functions.
   Turán [13] shows that any nontrivial graph property has sensitivity $\Omega(\sqrt{n})$, and asks whether the same would be true for any boolean function on n variables which is invariant under a transitive group of permutations. This is a slightly weaker version of the question above.
3. We show that $bs_2(f) \leqslant 2s^0(f)s^1(f)$. Rubinstein's construction [10] gives an example where $bs_2(f)=s^0(f)s^1(f)/2$. Can this gap be tightened any further? (Note that an improvement to the constants in Theorem 3.1 might lead to a subexponential upper bound on block sensitivity in terms of sensitivity.)

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