2015 Chicago Math REU Apprentice Program Exercises
Friday, July 3

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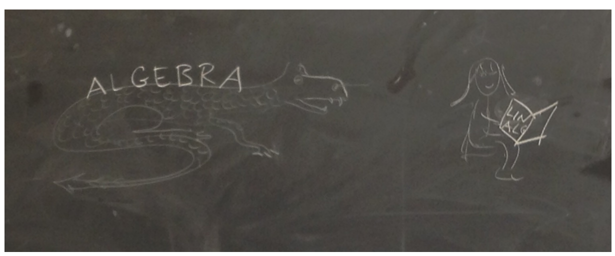
Exercise 10.1. If $S \subseteq V$ and T is a maximal linearly independent subset of S, then $\mathrm{Span}(S) = \mathrm{Span}(T)$.

Exercise 10.2. (Dragon Problem) Algebra Dragon visits Princess at midnight. (Exceptionally, she is up because she is preparing for a Linear Algebra test the next morning.) Dragon gives Princess a 12×20 matrix of rank 8 and tells her to change exactly one entry at noon every day. Dragon will visit again at midnight every night (while Princess is asleep) and change one entry. If the rank of the matrix ever drops to 7 or below, Dragon will eat Princess. Can you help Princess?



Exercise 10.3. ♡ Given any nonzero real polynomial f, there exists an nonzero real polynomial g such that $f \cdot g$ is a prime-exponent polynomial (i.e., every exponent is a prime number, like $5x^2 + 6x^5 - 2x^{53}$).

Exercise 10.4. Let φ be Euler's phi function, i.e., for a positive integer n, the value $\varphi(n)$ is the number of those integers j in the interval $1 \le j \le n$ that are relatively prime to n.
(a) Show: if p is a prime then $\varphi(p^k) = p^k - p^{k-1} = p^k(1 - 1/p)$.
(b) Show that if $\gcd(a, b) = 1$, then $\varphi(ab) = \varphi(a)\varphi(b)$.
(c) If $n = p_1^{k_1} \cdots p_s^{k_s}$, then $\varphi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_s})$.

Exercise 10.5. (a) Prove:

$$\sum_{d|n} \varphi(d) = n.$$

(b)♡ Let $A = (a_{ij})_{n \times n}$ be the matrix defined by $a_{ij} = \gcd(i, j)$. Prove that

$$\det(A) = \varphi(1)\varphi(2) \cdots \varphi(n).$$

Exercise 10.6. Let S_n be the set of permutations of the set $\{1, \ldots, n\}$.

Exercise 10.7. Prove that multiplying a permutation by a transposition changes the number of inversions by an odd number.

Exercise 10.8. What is the probability of that a random permutation of S_n is an n-cycle? (Find $\mathbb{P}_{\pi \in S_n}[\pi \text{ is an n-cycle }]$.)

Exercise 10.9. Prove that as $n \to \infty$,

$$\mathbb{P}_{\pi \in S_n}[\ \pi \text{ is fixed-point-free}] \to \frac{1}{\mathrm{e}}.$$

Exercise 10.10. * Let a_n denote the number of fixed-point-free odd permutations in S_n and let b_n denote the number of fixed-point-free even permutations in S_n. Is $a_n \lesseqgtr b_n$?

* Send email to the instructor if you work on this problem.