2015 Chicago Math REU Apprentice Program Exercises
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Exercise 23.1 (Exercise 21.9)**.** Let $\mathbb{F}$ be any field. Let $U \leq \mathbb{F}^n$. Show that $(U^\perp)^\perp = U$.

Proof. $\supseteq$ easy. In fact, $(S^\perp)^\perp \supseteq S$ for any $S \subset \mathbb{F}^n$.

Now, it suffices to show that $\dim U = \dim\big((U^\perp)^\perp\big)$. Let $d = \dim U$. By Rank-Nullity, $\dim U^\perp = n - d$. Again by Rank-Nullity, $\dim(U^\perp)^\perp = n - (n-d) = d$. $\square$

Exercise 23.2 (Exercise 21.10 -Berlekamp)**.** Under Eventown rules, a town of n citizens cannot have more than $2^{\lfloor n/2 \rfloor}$ clubs.

Proof. Let $A \subseteq \{1 \dots n\}$. Define the *incidence vector* $v_A \in \mathbb{R}^n$ by $(v_A)_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$.
Let $A, B \subseteq \{1 \dots n\}$. We notice that $v_A \cdot v_B = |A \cap B|$.

The Eventown conditions say that any two (possibly nondistinct) clubs have even intersection, i.e., $(\forall i, j)(v_i \cdot v_j = \text{even})$. Over $\mathbb{F}_2$, this is equivalent to saying $(\forall i, j)(v_i \cdot v_j = 0)$. We consider a collection of distinct clubs in $\{1 \dots n\}$, encoded as a set of incidence vectors, $\mathcal{C} = \{v_1 \dots v_m\} \subset \mathbb{F}_2^n$. The clubs satisfy Eventown rules iff $\mathcal{C}$ is a totally isotropic *subset*, i.e., $\mathcal{C} \perp \mathcal{C}$, i.e., $\mathcal{C} \subseteq \mathcal{C}^\perp$.

Let now $\mathcal{C}$ be a maximal Eventown system.
Claim: $\mathcal{C}$ is a subspace of $\mathbb{F}_2^n$.
Proof: If $\mathcal{C} \perp \mathcal{C}$, then $(\operatorname{span}\mathcal{C}) \perp (\operatorname{span}\mathcal{C})$, so for $\mathcal{C}$ to be maximal we must have $\mathcal{C} = \operatorname{span}\mathcal{C}$.

Now to complete the proof of the Eventow Theorem, we note that $\mathcal{C}$ is a totally isotropic subspace; so it has dimension $d \leq \lfloor n/2 \rfloor$ and therefore $|\mathcal{C}| = 2^d \leq 2^{\lfloor n/2 \rfloor}$ as claimed. $\square$

Exercise 23.3. This upper bound is tight.

To show this, we need to find an Eventown club system of $2^{\lfloor n/2 \rfloor}$ clubs. 'Married couples solution:' split the citizens into $\lfloor n/2 \rfloor$ pairs and form all clubs possible that keep the couples together. (If n is odd, this solution leaves one person complete out.)

Exercise 23.4 (No one left out)**.** For $n \geq 7$, find an optimal club system such that every citizen is a member of at least one club. (Hint: it suffices to find a basic of the space of clubs, i.e., a totally isotropic set of $\lfloor n/2 \rfloor$ clubs that are linearly independent and involve every citizen.)

Proof. For n even, the married couple solution is fine.

For $n = 7$, we can find 3 clubs each with 4 members, and each pair sharing 2 members. (Look at the faces of a cube that contain a particular vertex.)

For $n = 9$, take the 3 linearly independent clubs on $\{1 \dots 7\}$ then the club containing $\{8, 9\}$. This is a set of 4 clubs satisfying the hint. This extends to larger odd n. $\square$

Exercise 23.5. If $U \leq \mathbb{F}_2^n$ is a maximal totally isotropic subspace then $\dim U = \lfloor n/2 \rfloor$.

Remark 23.6. One can generalize these concepts to more general bilinear inner products over fields. The central result of the subject is called "Witt's theorem" which tells the dimensions of maximal totally isotropic subspace in this more general context. Emil Artin's "Geometric Algebra" is an accessible text on the subject.

23.1 Spectral Theorem

Theorem 23.7. *If $A \in M_n(\mathbb{R})$ and $A = A^T$, then there exists an orthonormal eigenbasis of A.*

Equivalently: Let V be a finite-dimensional Euclidean space. Let $\phi : V \to V$ be a symmetric linear transformation, i.e., $(\forall x, y)(\langle \phi x, y \rangle = \langle x, \phi y \rangle)$. Then, ϕ has an orthonormal eigenbasis (i.e., an ONB of V consisting of eigenvectors of ϕ).

The equivalence follows from Exercise 22.1: Let $\underline{b}$ be an orthonormal basis in the n-dimensional Euclidean space V and let $\phi \in \mathrm{Hom}(V, V)$. Prove that ϕ is a symmetric transformation iff $[\phi]_{\underline{b}}$ is a symmetric matrix.

Lemma 23.8. *If $\phi : V \to V$ is symmetric, then there exists an eigenvector.*

Definition 23.9. Recall that U is *ϕ-invariant* if $(\forall x \in U)(\phi(x) \in U)$, i.e., $\phi(U) \subseteq U$.

Lemma 23.10. *Suppose that $\phi : V \to V$ is symmetric. If $W \leq V$ and W is ϕ-invariant, then $W^\perp$ is ϕ-invariant.*

Remark 23.11. The symmetric assumption on the transformation cannot be omitted. A counterexample is given by a 'shearing' map, given by the matrix $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$. Another assumption on the transformation that suffices is orthogonality. Recall that $\psi : V \to V$ is called an *orthogonal transformation* if $(\forall x, y \in V)(\langle x, y \rangle = \langle \psi x, \psi y \rangle)$. (These are the 'congruences.')

Exercise 23.12. (Exercise 22.2) Suppose that $\psi : V \to V$ is orthogonal. If $W \leq V$ and W is ψ-invariant, then $W^\perp$ is ψ-invariant.

Recall that any linear transformation of a real vector space of odd dimension must have a real eigenvalue and therefore a real eigenvector.

Exercise 23.13. For every even n find an $n \times n$ orthogonal matrix with no real eigenvalue.

Solution: $M_2(\mathbb{R})$, the rotation matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ works. In $M_4(\mathbb{R})$, the block-diagonal matrix $R(\theta, \sigma) = \begin{pmatrix} R_\theta & 0 \\ 0 & R_\sigma \end{pmatrix}$ will work. Why?

Exercise 23.14. Let A be a block-diagonal matrix with square diagonal blocks B_i. Then the characteristic polynomial of A is the product of the characteristic polynomials of the B_i.

$$A = \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & B_k \end{pmatrix}$$

Exercise 23.15. Let A be a block-triangular matrix (as above except the blocks above the diagonal are arbitrary). Prove: even under this more general condition, the characteristic polynomial of A is the product of the characteristic polynomials of the B_i.

Exercise 23.16. Let M be a block-tringualr matrix with 2×2 blocks: $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. The $\det(M) = \det(A)\det(B)$. Use this fact to prove that $\det(RS) = \det(R) \cdot \det(S)$.

23.2 Existence of a Real Eigenvector for a Symmetric Matrix

Exercise 23.17. Let $f(t) = \frac{a+bt+ct^2}{d+et^2}$, where $d \neq 0$. Suppose that $f(0) = \max_t f(t)$. Show that $b = 0$.

Let $\phi : V \to V$ be a symmetric transformation. Define the Rayleigh quotient $R_\phi : V \setminus \{0\} \to \mathbb{R}$ by $R_\phi(x) = \frac{\langle x, \phi x \rangle}{||x||^2}$.

Definition 23.18. A subset $D \subseteq \mathbb{R}^n$ is *bounded* if there exist $C < \infty$ such that $(\forall x \in D)(||x|| \leq C)$. A subset $D \subseteq \mathbb{R}^n$ is *closed* if for all $x_1, x_2, \ldots \in D$ with limit point $x = \lim_{n\to\infty} x_n$ it holds that $x \in D$. A subset $D \subseteq \mathbb{R}^n$ is *compact* if D is closed and bounded.

Theorem 23.19. *Suppose $D \subseteq \mathbb{R}^n$ is compact and $f : D \to \mathbb{R}$ is continuous. Then, there exists x_0 such that $f(x_0) = \max_{x \in D} f(x)$.*

Exercise 23.20. There exists $x_0 = \operatorname{argmax} R_\phi(x)$, i.e. $(\forall x \in V \setminus \{0\})(R_\phi(x_0) \geq R_\phi(x))$

Theorem 23.21. *If $x_0 = \operatorname{argmax}_x R_\phi(x)$, then x_0 is an eigenvector of ϕ.*

Proof. Let $U = (\text{span}(x_0))^\perp$. Take any $u \in U$ with $u \neq 0$, so that $u \perp x_0$. Consider

$$\begin{aligned} R_\phi(x_0 + tu) &= \frac{\langle x_0 + tu, \phi(x_0 + tu)\rangle}{||x_0 + tu||^2} \\ &= \frac{\langle x_0, \phi x_0\rangle + t\langle x_0, \phi u\rangle + t\langle u, \phi x_0\rangle + t^2\langle \phi u, \phi u\rangle}{||x_0||^2 + t^2||u||^2} \\ &= \frac{\langle x_0, \phi x_0\rangle + 2t\langle u, \phi x_0\rangle + t^2\langle u, \phi u\rangle}{||x_0||^2 + t^2||u||^2} \\ &= \frac{a + bt + ct^2}{d + et^2}, \end{aligned}$$

for appropriately chosen a, b, c, d, e. But, $R_\phi(0) = \max f(t)$, so $b = 2\langle u, \phi x_0\rangle = 0$, so $u \perp x_0$. Conclusion so far: $(\forall u \in V)(u \perp x_0 \Rightarrow u \perp \phi x_0)$.

Therefore, $\phi x_0 \perp \langle x_0\rangle^\perp$ so $\phi x_0 \in \langle x_o\rangle^{\perp\perp} = \langle x_0\rangle$. Therefore, there exists λ such that $\phi x_0 = \lambda x_0$, and x_0 is an eigenvector. $\square$

Recall that the matrices $A, B \in M_n(\mathbb{F})$ are *similar* if $(\exists C \in M_n(\mathbb{F}))(A = C^{-1}BC)$.

Definition 23.22. $A, B \in M_n(\mathbb{R})$ are *orthogonally similar* if $(\exists C \in O(n))(A = C^{-1}BC)$. (Recall: $O(n)$ is the set of $n \times n$ orthogonal matrices.)

Recall that A is diagonalizable if A is similar to a diagonal matrix.

Definition 23.23. $A \in M_n(\mathbb{R})$ is orthogonally diagonalizable if A is orthogonally similar to a diagonal matrix.

Exercise 23.24. $\underline{e} = (e_1, \ldots, e_n)$ is an eigenbasis of the linear transformation $\phi : V \to V$ if and only if the matrix $[\phi]_{\underline{e}}$ is diagonal.

Theorem 23.25. *Let $A \in M_n(\mathbb{R})$. A is symmetric $\iff$ A is orthogonally similar to a diagonal matrix.*

Exercise 23.26.

(a) Show that the $\Rightarrow$ direction of Theorem 23.25 is equivalent to the Spectral Theorem.

(b) Prove the $\Leftarrow$ of Theorem 23.25.