2015 Chicago Math REU Apprentice Program Exercises
Wednesday, June 24

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Exercise 3.1. Verify that $\mathbb{F}_p$ is a field.

Exercise 3.2. Prove that 1, $\sqrt{2}$, and $\sqrt{3}$ are linearly independent as elements of $\mathbb{R}$ when $\mathbb{R}$ is regarded as a vector space over $\mathbb{Q}$.

Exercise 3.3. Let V be a vector space. Show that a linear transformation $T : V \to V$ is uniquely determined by the image under T of the basis vectors of V, i.e., if $\{v_i\}_{i \in I}$ is a basis for V and $S : V \to V$ is a linear transformation with $T(v_i) = S(v_i)$ for all $i \in I$, then $S(v) = T(v)$ for all $v \in V$.

Exercise 3.4. Let μ and T be defined as in class, so $T : \mathbb{R} \to \mathbb{R}$ is a linear transformation of $\mathbb{R}$ regarded as a $\mathbb{Q}$ vector space, and if Q is an $a \times b$ rectangle, $\mu(Q) = T(a)T(b)$. Assume that a rectangle R is subdivided into rectangles $Q_1, \ldots, Q_n$. Show that $\mu(R) = \sum_{i=1}^{n} \mu(Q_i)$.

Exercise 3.5. Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix. Show that $\lambda \in \mathbb{R}$ is an eigenvalue of M if and only if $\det(\lambda I - M) = 0$.

Exercise 3.6. Prove that elementary row and column operations on a matrix preserve row and column rank.

Exercise 3.7. Given $\lambda \in \mathbb{R}$, and $T : V \to V$ a linear transformation, define

$$U_\lambda = \{v \in V \mid Tv = \lambda v\}.$$

Show that U_λ is a vector space.

Exercise 3.8. Show that the algebraic multiplicity of an eigenvalue of a matrix is at least the geometric multiplicity of that eigenvalue.