Day 3 (PS)

Wednesday, June 21, 2017 9:38 AM

6.34 (a) Find the determinant of

$$\begin{bmatrix} \alpha & \beta & \beta & \cdots & \beta \\ \beta & \alpha & \beta & \cdots & \beta \\ \beta & \beta & \alpha & \cdots & \beta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta & \beta & \beta & \cdots & \alpha \end{bmatrix} \quad \begin{bmatrix} \alpha & \beta & \beta & \cdots & \beta \\ \beta-\alpha & \alpha-\beta & 0 & \cdots & 0 \\ 0 & \beta-\alpha & \alpha-\beta & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & \beta-\alpha & \alpha-\beta & 0 \\ 0 & \cdots & 0 & \beta-\alpha & \alpha-\beta \end{bmatrix}$$

$$\alpha(\alpha-\beta)^{n-1} + \beta \sum_{i=1}^{n-1} (-1)^i (\beta-\alpha)^i (\alpha-\beta)^{n-2-i}$$

$$\sum_{i=1}^{n-1} (-1)^i \left[(-1)^i (\alpha-\beta)^i\right] (\alpha-\beta)^{n-2-i}$$

$$(n-1)(\alpha-\beta)^{n-1}\beta = \boxed{(\alpha+(n-1)\beta)(\alpha-\beta)^{n-1}}$$

(b) $$A_n = \begin{bmatrix} 1 & 1 & & & 0 \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & & & \ddots & \\ 0 & & & 1\;1 & 1 \\ & & & 1 & 1 \end{bmatrix}$$

$C_{11} = \det A_{n-1}$

$C_{21} = \det A_{n-2} \longrightarrow$ $$\begin{bmatrix} 1 & 1 & 0 & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$|1| = 1$

$$\boxed{\det(A_n) = \det(A_{n-1}) - \det A_{n-2}}$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

1

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

1
0
−1
−1
0
1
1
0
−1
⋮

(c) same as above, but

$$\det C_{21} = -\det A_{n-2}$$

and $\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$, so

$$\det A_n = \det A_{n-1} + \det A_{n-2}$$

1, 2, 3, 5, 8, 13, —

6.3.5 Vandermonde.

multiply by α_1 and substract

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \alpha_2 - \alpha_1 & & & & \\ 0 & \alpha_2^2 - \alpha_2\alpha_1 & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}$$

$$\alpha^{n-1} - \alpha_2^{n-2}\alpha_1 \quad \cdots \quad \alpha_n^{n-1} - \alpha_n^{n-2}\alpha_1$$

$$\alpha_2^{n-1} - \alpha_2^{n-2}\alpha_1 \quad \cdots \quad \alpha_n^{n-1} - \alpha_n^{n-2}\alpha_1$$

$$\det = \det \begin{bmatrix} \alpha_2 - \alpha_1 \\ \alpha_2^2 - \alpha_2\alpha_1 \\ \\ \\ \end{bmatrix}$$

$$\begin{matrix} \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \cdots & \alpha_n - \alpha_1 \\ \alpha_2(\alpha_2 - \alpha_1) & \alpha_3(\alpha_3 - \alpha_1) & & \vdots \\ \alpha_2^2(\alpha_2 - \alpha_1) & \alpha_3^2(\alpha_3 - \alpha_1) & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_2^{n-2}(\alpha_2 - 1) & \alpha_3^{n-2}(\alpha_3 - \alpha_1) & & a_n^{n-2}(a_n - \alpha_1) \end{matrix}$$

factor out one from each column.

$$\downarrow$$

$$\prod_{j=2}^{n} (\alpha_j - \alpha_1) \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_2 & \alpha_3 & \cdots & & & \alpha \\ \vdots & & & & & \vdots \\ \alpha_2^{n-2} & \cdots & & & & \alpha_n^{n-2} \end{bmatrix}$$

$$= \boxed{\prod_{j<i} (\alpha_i - \alpha_j)}$$

If A is integer matrix, when is A^{-1} integer?

If $\det A = \pm 1$,

$$\operatorname{adj}(A) = (C_{ij})^T$$

$$= (-1)^{i+j} \det({}_iA_j)$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$$

$A = \det(A)$

A^{-1} must be integers since

$\operatorname{adj}(A)$ composed of cofactors.

(addition / multiplication)

If A^{-1} is integer matrix,

$\det A \det A^{-1} = \det I = 1$

$\det A$ and $\det A^{-1}$ must be integral.

so $1, 1$ $-1, -1$ are only pairs

that are multiplicative inverses.

If $f : \mathbb{R}^M \to \mathbb{R}$ is linear, then $\exists$

$\vec{a} \in \mathbb{R}^M$ s.t. $f(\vec{x}) = \vec{a} \cdot \vec{x}$. $\forall \vec{x} \in \mathbb{R}^M$.

$$\vec{x} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{e}_2} + \dots + x_m \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\vec{e}_M}$$

$$f(\vec{x}) = f(x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_m \vec{e}_m)$$

$$= f(x_1 \vec{e}_1) + f(x_2 \vec{e}_2) + \dots + f(x_m \vec{e}_m)$$

$$= x_1 f(\vec{e}_1) + x_2 f(\vec{e}_2) + \dots + x_m f(\vec{e}_m)$$

Let $\vec{a} = \langle f(\vec{e}_1), f(\vec{e}_2), \dots, f(\vec{e}_m) \rangle$.

$$\text{sgn}(\sigma) = \begin{cases} 1 & \text{if even \# of transpositions} \\ -1 & \text{if odd} \end{cases}$$

product of

transposition - swap value of two positions

sgn is well-defined (Hard to show).

WTS: $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$.

$1 \to 2 \to 1$

$2 \to 3 \to 2$

$3 \to 1 \to 3$

$\sigma = \tau_1 \tau_2 \cdots \tau_n$ $\quad \sigma^{-1} = \tau_n \cdots \tau_2 \tau_1$

Transpositions are their own inverses

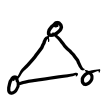
$(\tau_i^2 = I \quad \forall i \in [n])$

so $\sigma\sigma^{-1} = I$.

\# of transpositions same, so same sgn. $\square$

If G is a triangle-free graph then $m \le \frac{n^2}{4}$.

$K_3 \notin G$

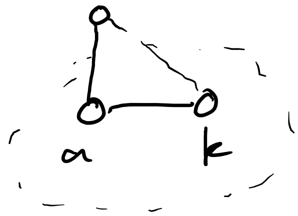


Induction?

$n = 1$ ✓ $\quad m = 0 \le \frac{1}{4}$

$n = 2$ ✓ $\quad m \le 1 \le 1$

$n \le k-1 \rightarrow n \le k$



$k-2$

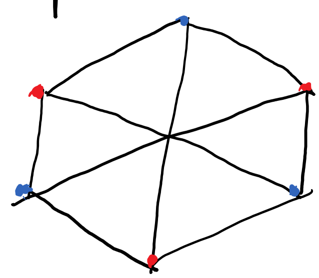
$\le (k-2)+1 = k-1$

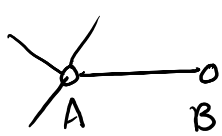
$\le \frac{(k-2)^2}{4}$

$\frac{k^2 - 4k \cancel{+4}}{4} + \frac{4k \cancel{-4}}{4}$

$\le \frac{k^2}{4}$ $\square$

n even - complete bipartite graph works.





$\deg(A) + \deg(B) \le n$

($n-2$ other connections, $A \to B$, $B \to A$)

x x x x x

$n-2$ pts.

$$\sum_{\{A,B\} \in E} (\deg(A) + \deg(B)) \le nm$$

$$\sum_{A \in V} \deg(A)^2 =$$

each vertex appears deg A times in the sum ...

and we sum (deg A)s, so

$\boxed{\text{HW}}$ Finish this proof.

$$\sum_{A \in V} (\deg A)^2 = \sum_{\{A,B\} \in E} (\deg(A) + \deg(B)).$$

<u>Tree</u> Connected, acyclic graph

"forest"

$\forall$ trees T with $n \geq 2$ has a vertex of degree 1 (leaf).

Can't have $\deg(i) = 0$, for some $i \in V$ if $n \geq 2$ — not connected

If $\nexists\, i \in V$ s.t. $\deg(i) = 1$,

$\forall\, i \in V$ $\deg(i) \geq 2$.

Pick a point and start walking — don't reuse edges. will eventually revisit a point (cycle) or terminate (vertex of deg 1?)

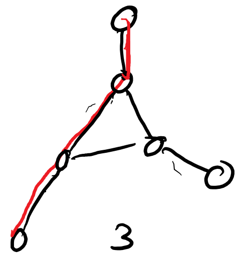
<u>Maximum path</u> — A path with the largest number of vertices.

...path — A path which cannot ... path

longer

be extended to a

Every path between vertices of degree 1 is maximal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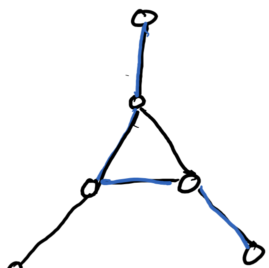
Converse true for trees, not other way around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Proof. Let T be a tree with $n \geq 2$. Take a maximal path in T.

(Since T is connected and $n \geq 2$, $m \geq 1$.)

If the end points had deg ≥ 2, it would have to connect to another pt. It cannot be connected to other pts in graph (would form a cycle), so must be separate → then can extend path. Contradiction.

$\exists$ two vertices of degree 1.

path - no repetition.

walk - allowed to revisit

vertices has $n-1$ edges.

A tree with n vertices has n-1 edges.

with 2 vertices, both have deg 1 - 1 edge.

1 vertex — 0

n+1 vertex tree has at least 1 vertex of deg. 1.

remove it

removed vertex cannot (dis)connect anything (only 1 edge) and removing a vertex cannot create a cycle

n vertex tree has n-1 edges by inductive hyp

add back in.

n+1 vertex tree has n edges. $\square$

G is connected iff G has a spanning tree

G is connected $\Leftarrow$ G has a spanning tree.
trivial

connected $\Rightarrow$ G has a spanning tree
... vertex

Since connected, $\exists$ path from every to every other vertex.

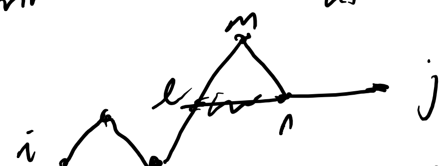
Take union of all paths - remove one cycle edge (redundant)

If G has 0 cycles, tree

If G has n cycles, it has a spanning tree.

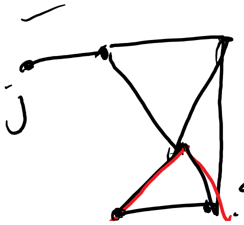
G has n-1 cycles, then if we remove 1 edge from a cycles there will be at least 1 fewer cycle

Graph still connected (two ways to get around l->n)



By strong induction, tree exists.

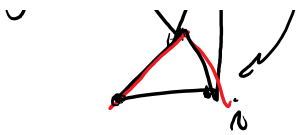
Two vertices are connected by a walk iff they are connected by a path



<u>Maximal tree</u>

Assume maximal tree is not spanning. $\hookrightarrow T$

$\exists$ path



Assu... $\hookrightarrow T$

spanning.

Since G is connected $\exists$ path between $i \in T$ and $j \notin T$.

Extend the tree T by adding the first edge on the path from j to i that intersects T. Then T is not maximal

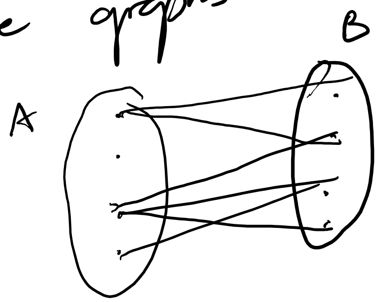
$\rightarrow$ contradiction

Thus every maximal tree is spanning.

A maximal tree exists.

$\therefore$ A spanning tree exists

Bipartite graphs:



A graph is bipartite iff it has no odd cycles.

Bipartite $\Rightarrow$ no odd cycles

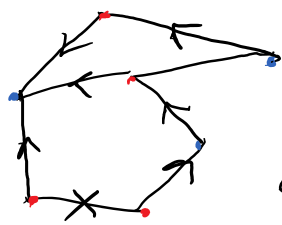
$\exists$ odd cycle, there must exist an ... not bipartite.

edge $A \to A$ or $B \to B$

No odd cycles $\Rightarrow$ bipartite.

Take a walk along the paths, alternating colors - if conflict, you have an odd cycle

alternate color walk



every node has a parent - unique path to each node.

can have conflict edges.