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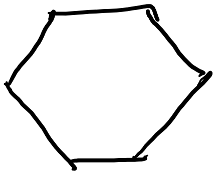
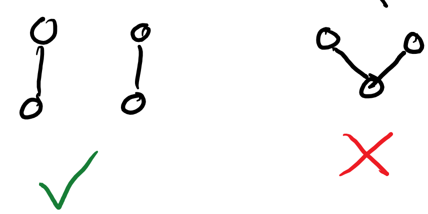
Matching in a graph : set of disjoint edges

__matching number__ $\nu(G)$ = max # of edges in a matching

\nu

(not v)

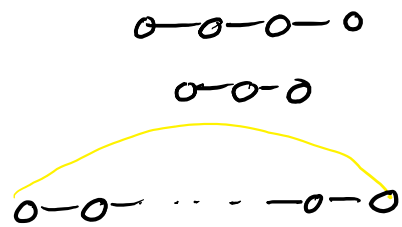


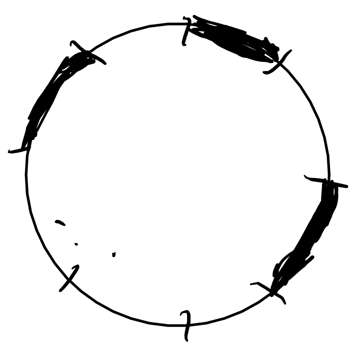


$\nu(C_n) = \lfloor \frac{n}{2} \rfloor$ (floor of $\frac{n}{2}$)

$\nu(P_n) = \lfloor \frac{n}{2} \rfloor$

path of length $n-1$





Let M be a __maximal__ matching. (Obviously $|M| \le \nu(G)$).

DO $\exists\, G, M$ s.t. $\frac{\nu(G)}{|M|} = 2$.

DO $(\forall k)(\exists G)(\nu(G) = 2|M| = 2k$ and G is connected$)$

$\boxed{\text{HW}}$ $\nu(G) \leq 2|M|$ for all maximal matchings M.

Linear combinations.

$\mathbb{R}^n$

$v_1, \ldots, v_k \in \mathbb{R}^n$

Def. A <u>linear combination</u> of the v_i is an expression of the form

$\alpha_1 v_1 + \cdots + \alpha_k v_k$ where $\alpha_i \in \mathbb{R}$

coefficients "scalars"

α_i is the coefficient of v_i.

could be infinite

$S \subseteq \mathbb{R}^n$.

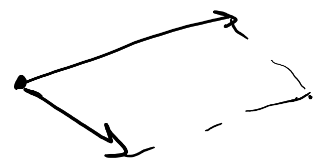
Span(S) = set of all linear combinations of S.

Linear combinations of an infinite set := linear combinations of a finite subset.

$\text{Span}(1, x, x^2, x^3, \dots) =$ all polynomials <u>over $\mathbb{R}$</u>

coefficients are real.

$$= \sum_{i=0}^{n} \alpha_i x^i$$

0 

G_2 2D- geometry

G_3 3D - geometry

Span of 2 vectors in G_3 : plane

(almost always .. except when parallel — a line — and when both are 0 — the origin).

$\forall S \subseteq \mathbb{R}^n, \quad 0 \in \text{Span}(S)$.

A <u>trivial</u> linear combination is one in which all coefficients are 0.

The value of this l.c. is $\vec{0}$.

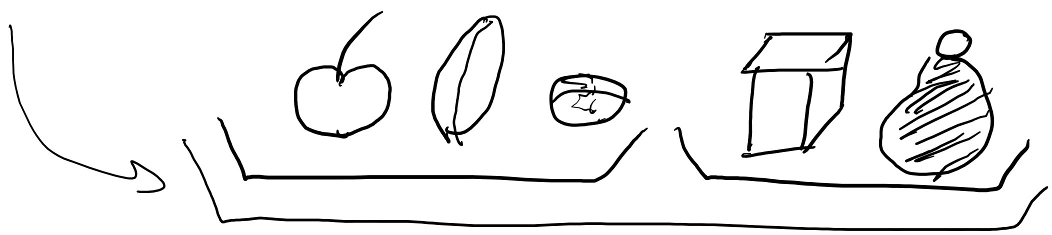
What is $\text{span}(\emptyset)$? $\emptyset$? $\{0\}$?

sum of weights of some objects.

$= S_1$

$= \underline{\underline{0}}$.

$S_3 = S_1 + S_2$



If S_2 = empty sum...

$S_3 = S_1 + S_2$

we want to preserve $S_1 = S_3$, so $\boxed{S_2 = 0}$.

What if addition doesn't make sense across elements?

Sum of empty set of bananas ...

0 bananas.

Thus, $\mathrm{Span}(\emptyset) = \{0\}$, and

$\forall S \subseteq \mathbb{R}^n$, $0 \in \mathrm{Span}(\emptyset)$.

What about 3 vectors?

Usually the span is the whole space...

unless...

- all three are coplanar – a plane (cannot escape it)
- all three are parallel – a line
- all three are 0 – the origin (a point).

If three are not coplanar there is a unique way to reach each point.

__Def.__ $v_1, v_2, \ldots, v_k$ are __linearly independent__ if __only__ their trivial linear combination is zero.

If $L_1 \subseteq L_2$ and L_2 is linearly independent
sublist
then L_1 is linearly independent.

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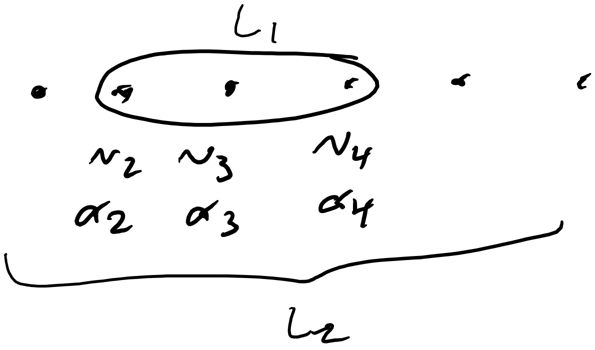

suppose L_1 is not linearly independent

Then $\alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$ nontrivially

(at least one of $\alpha_2, \alpha_3, \alpha_4$ is nonzero)

and $\sum_{i=1}^{6} \alpha_i v_i = 0$ with $\alpha_1, \alpha_5, \alpha_6 = 0$, $\square$

so L_2 is not linearly independent

$[v]$ is linearly <u>dependent</u> $\iff \vec{v} = \vec{0}$.

$$\alpha \vec{v} = \vec{0} \quad \forall \alpha \neq 0$$

$[v, v]$ is linearly dependent:

$$1 \cdot \vec{v} + (-1) \cdot \vec{v} = \vec{0}$$

<u>Cor.</u> If L is linearly independent, then

(i) $0 \notin L$. (0 is sublist + linearly dependent)

(ii) no two vectors on the list are equal.

([v, v] is sublist + linearly dependent)

What about for 3 vectors?

$\vec{a}, \vec{b}, \vec{a}+\vec{b}$

$1 \cdot \vec{a} + 1 \cdot \vec{b} + (-1)(\vec{a}+\vec{b}) = \vec{0}$.

Proving linear independence is harder - show any possible combination cannot be 0.

Review: A square matrix is <u>singular</u> if its determinant is 0.

<u>Thm</u> Let $A \in M_n(\mathbb{R})$. Then

A is nonsingular $\iff$ its columns are linearly independent.

<u>Proof</u> Equivalent statement:

A is singular $\iff$ columns are linearly dependent.

<u>Lemma</u> A list $[v_1, \ldots, v_k]$ is linearly dependent iff $(\exists i)(v_i \in \mathrm{Span}(v_1, \ldots \underset{i}{\wedge} \ldots v_k))$

i^{th} is missing.

Proof.

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$\Leftarrow$

$$v_i = \sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j v_j$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j v_j - v_i = 0$$

$\boxed{-1 \neq 0}$

(nontrivial)

$\Rightarrow$ $\exists$ coefficients s.t.

$\sum \alpha_j v_j = 0$, not all

α_j are 0.

$\therefore (\exists i)(\alpha_i \neq 0)$

$$\alpha_i v_i = \sum_{j \neq i} -\alpha_j v_j$$

$$v_i = \sum_{j \neq i} -\frac{\alpha_j}{\alpha_i} v_j .$$

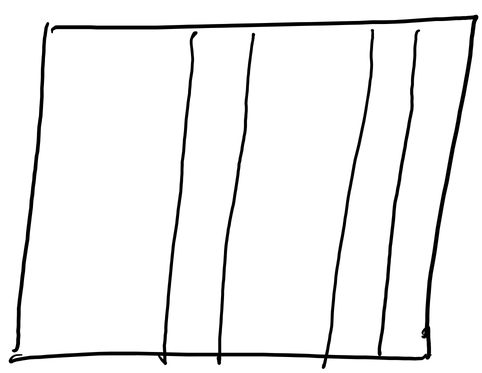
(Lin comb.) $\square$

Lin. dependent $\Rightarrow$ singular.

$\exists i \quad v_i = \sum_{j \neq i} \alpha_j v_j$

Desired Conclusion (DC):

$\det A = 0$



$v_i \quad -\alpha_j v_j = 0.$

A sequence of elementary column operations turns v_i into the 0 vector.

Thus, $\det A = 0$.

singular $\Rightarrow$ lin. dependent or

lin independent $\Rightarrow$ nonsingular. ($\det \neq 0$)

<u>Lemma.</u> $v_1, \dots, v_k$ is

linearly independent iff

$\underbrace{v_1', \dots, v_k'}$ is linearly independent,

where

result of elementary operation (i, ℓ, λ)

$$v_j' = \begin{cases} v_j & j \neq i \\ v_j - \lambda v_\ell & j = i \end{cases}$$

$v_i' = v_i - \lambda v_\ell$.

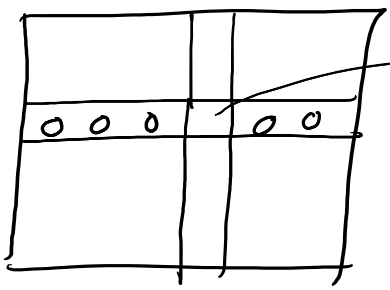
It suffices to prove one direction, since elementary operations are invertible.

The inverse of (i, ℓ, λ) is $(i, \ell, -\lambda)$.

<u>Proof</u> Equiv. to.

$v_1, v_2, \dots v_k$ lin. dep. $\Leftrightarrow$ $v_1', v_2', \dots, v_k'$ lin. dep.

DO Finish the proof.

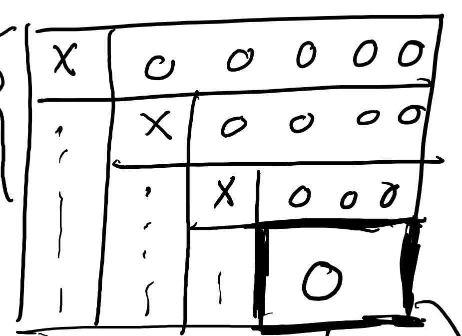


$a_{ij} \neq 0$ (pivot) $\Rightarrow$ kill i^{th} row using row operations

Lemma: Permuting rows does not affect the linear independence of columns. DO

(obviously same for columns.)

k



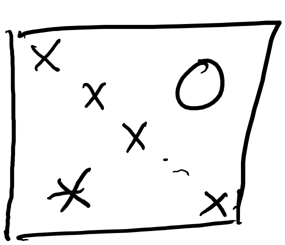
$n-k$

what if we eventually get stuck? all zeros left

$n-k$

Then columns are linearly dependent ($\exists$ zero column). This is a contradiction, since we started with a linearly independent matrix and lin. indep. is preserved by our operations.

Thus, we cannot stop until $n = k$.



Lower triangular matrix: $\det = \prod$ pivots

$\det(\text{original}) = \pm \prod$ pivots (by row/column operations + permutations)

$\neq 0$.

$\square$

$\boxed{\text{CH}}$ Find a continuous curve in $\mathbb{R}^n$

$$f: \mathbb{R} \to \mathbb{R}^n$$

s.t. its points are in <u>general position</u>, meaning every n of them are linearly independent.

(i.e. $\forall a_1 < a_2 < \dots < a_n$, $[f(a_1), \dots, f(a_n)]$ linearly independent)

DO Above for $\mathbb{R}^2$.

$$f: \mathbb{R} \to \mathbb{R}^n$$

$t \in \mathbb{R}$ $\quad f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix} \to$ get <u>simple</u> formulas for f_i.

Systems of Linear Equations $\quad A = (a_{ij})_{k \times n}$

$$\left.\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \vdots \\ a_{k1}x_1 + \dots + a_{kn}x_n &= b_k \end{aligned}\right\} \overset{\text{DO}}{\Longleftrightarrow} A\underline{x} = \underline{b}$$

$\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} \in \mathbb{R}^k$

$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$A = [\underline{a_1}, \ldots, \underline{a_n}]$ where $\underline{a_j}$ is j^{th} column.

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{kj} \end{pmatrix} \in \mathbb{R}^k$$

DO $A\underline{x} = x_1 \underline{a_1} + \cdots + x_n \underline{a_n}$ (matrix multiplication)

$b = x_1 \underline{a_1} + \cdots + x_n \underline{a_n} \iff A\underline{x} = \underline{b}$

Thm $A\underline{x} = \underline{b}$ is solvable $\iff$ $b \in \underbrace{\mathrm{Span}(\underline{a_1}, \ldots, \underline{a_n})}_{\text{column space of } A}$.

If $\underline{b} = \underline{0}$:

homogeneous system of linear equations

$A\underline{x} = \underline{0}$ is solvable – trivial solution always works.

Thm. Nontrivial solution exists $\iff$ columns are linearly dependent.

Thm $A \in M_n(\mathbb{R})$ is nonsingular $\iff$ $A\underline{x} = \underline{0}$ has no nontrivial solution.

DO $A \in M_n(\mathbb{R})$ is nonsingular $\iff$ rows are linearly independent.

Def. $\underline{x} \in \mathbb{R}^n$ is an <u>eigenvector</u> of $A \in M_n(\mathbb{R})$ with eigenvalue $\lambda \in \mathbb{R}$ if

$$A : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto Ax$$

(i) $\underline{x} \neq \underline{0}$

(ii) $A\underline{x} = \lambda \underline{x}$

Def. λ is an <u>eigenvalue</u> of $A \in M_n(\mathbb{R})$ if $\exists\ \underline{x} \neq 0$ in $\mathbb{R}^n$ s.t. $A\underline{x} = \lambda \underline{x}$.

(Eigenvectors cannot be 0, but eigenvalues can.)

Thm. $A \in M_n(\mathbb{R})$ is nonsingular $\Leftrightarrow$ 0 is not an eigenvalue.

$\boxed{\text{DO!}}$

$\boxed{\text{HW}}$ Eigenvectors to distinct eigenvalues are linearly independent,

i.e. if $v_1, \dots, v_k \in \mathbb{R}^n$, $v_1, \dots, v_k \neq 0$ and $A \in M_n(\mathbb{R})$ and $Av_i = \lambda_i v_i$, with λ_i all distinct, then $v_1, \dots, v_k$ are linearly indep.

Consider identity I.

$I\underline{v} = \underline{v}$.

All nonzero vectors are eigenvectors to eigenvalue 1.

If $\underline{v}$, λ are eigenvector/eigenvalue pair, then
$2\underline{v}$, λ also work:

$$A\underline{v} = \lambda\underline{v}$$

$$A \cdot 2\underline{v} = \lambda \cdot 2\underline{v}.$$

$\boxed{\text{HW}}$ If λ is an eigenvalue of A, then
λ^k is an eigenvalue of A^k.

$\boxed{\text{HW}}$ If $f \in \mathbb{R}[t]$, ← set of polynomials over $\mathbb{R}$
then $f(\lambda)$ is an eigenvalue of $f(A)$.

Note: If $f(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_k t^k$,
then (Def.) $f(A) = \alpha_0 I + \alpha_1 A + \dots + \alpha_k A^k$.

* Clarity matters!

Finding an eigenvector to a given eigenvalue amounts to finding a <u>nontrivial</u> solution to a <u>homogeneous</u> system of linear equations.

$$A\underline{x} = \lambda \underline{x} \qquad (\underline{x} \neq 0)$$
$$= \lambda I \cdot \underline{x}$$

$$\lambda I \cdot \underline{x} - A \cdot \underline{x} = 0$$

$$\underbrace{(\lambda I - A)}_{\text{characteristic matrix of } A}\underline{x} = \underline{0}$$

<u>Thm.</u> λ is an eigenvalue of $A \in M_n(\mathbb{R})$

$\iff (\lambda I - A)$ is singular

$\iff \boxed{\det(\lambda I - A) = 0}$ (no $\underline{x}$ anymore!)

characteristic polynomial of A:

$$f_A(t) := \det(tI - A)$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad tI - A = \begin{bmatrix} t - a_{11} & -a_{12} \\ -a_{21} & t - a_{22} \end{bmatrix}$$

$$\det(tI - A) = (t - a_{11})(t - a_{22}) - a_{12}a_{21}$$
$$= t^2 - \underbrace{\boxed{(a_{11} + a_{22})}}_{\text{Trace}(A)} t + \underbrace{\boxed{(a_{11}a_{22} - a_{12}a_{21})}}_{\det(A)}$$

$f(t) = \alpha_0 + \dots + \alpha_n t^n$

$f(0) = \alpha_0$

$\alpha_0 = \det(tI - A)\Big|_{t=0} = \det(-A) = (-1)^n \det A$

(multiplying each of n columns by -1 switches sign n times)

$A \in M_n(\mathbb{R})$, $A = (a_{ij})$

<u>Trace</u>: $Tr(A) = \sum a_{ii}$

DO $Tr(AB) = Tr(BA)$

$A \in \mathbb{R}^{k \times l}$ $B \in \mathbb{R}^{l \times k}$

DO $f_A(t)$. ~ $(A \in M_n(\mathbb{R}))$

(i) is a <u>monic</u> polynomial of degree n:

for $f_A(t) = \alpha_n t^n + \dots + \alpha_1 t + \alpha_0$, $\alpha_n = 1$.

(ii) $\alpha_{n-1} = -Tr(A)$

(iii) $\alpha_0 = (-1)^n \det A$

Question: What is the coefficient of t^{n-2}?

<u>Def.</u> A <u>basis</u> of $\mathbb{R}^n$ is a linearly independent list of vectors in $\mathbb{R}^n$ that <u>span</u> $\mathbb{R}^n$.

<u>Thm.</u> (later) "1st miracle of linear algebra"

If $B \subseteq \mathbb{R}^n$ is a basis then $|B| = n$.

(nontrivial)

$A \in M_n(\mathbb{R})$

<u>Def.</u> $v_1, \dots, v_n \in \mathbb{R}^n$ is an <u>eigenbasis</u> of A if $[v_1, \dots, v_n]$ is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Eigenvalues of triangular matrix

$$A = \begin{pmatrix} a_{11} & & X \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

$$f_A(t) = \det(tI - A)$$

$$= \det \begin{pmatrix} t - a_{11} & & -a_{ij} \\ & \ddots & \\ 0 & & t - a_{nn} \end{pmatrix}$$

$$= \prod (t - a_{ii})$$

$\therefore$ eigenvalues are a_{ii}.

<u>Thm.</u> λ is an eigenvalue of $A \iff f_A(\lambda) = 0$.

<u>Cor.</u> Distinct # of eigenvalues is $\leq n$.

$\boxed{\text{HW}}$ Prove: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has no eigenbasis.

DO Every linearly independent set in $\mathbb{R}^n$ extends to a basis.

DO <u>Cor.</u> Every set of n linearly independent vectors in $\mathbb{R}^n$ forms a basis.

$\boxed{\text{HW}}$ Let $A \in M_n(\mathbb{R})$ be triangular:

$$A = \begin{pmatrix} a_{11} & & X \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

suppose all diagonal elements are distinct.

Prove: A has an eigenbasis.

(≤ 4 lines using other exercises)

$\boxed{\text{HW}}$ Let $\alpha_1, \dots, \alpha_n$ be distinct reals.

$$f(t) = \prod (t - \alpha_i)$$

$$g_j(t) = \frac{f(t)}{t - \alpha_j} = \prod_{\substack{i=1 \\ i \neq j}}^{n} (x - \alpha_i)$$

Prove: $g_1, g_2, \dots, g_n$ are lin. independent

<u>Proof</u>: Suppose $\sum \beta_j g_j = 0$. <u>Show</u>: $\beta_1 = \dots = \beta_n = 0$.

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$\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathbb{R}^{\mathbb{N}}$: space of infinite sequences

$\underline{\alpha} = (\alpha_0, \alpha_1, \alpha_2 \ldots)$

$S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$

$S\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \ldots)$

left shift operator

$\boxed{\text{HW}}$

Find all eigenvalues and eigenvectors of S:

$S\underline{x} = \lambda \underline{x}$, $\underline{x} \neq (0, 0, 0, \ldots)$