3 DO exercises.

Monday, June 26, 2017 9:33 AM

Eigenvectors corresponding to distinct eigenvalues are linearly independent

λ_1 and λ_2 are distinct eigenvalues.

$\alpha_i \in \mathbb{R}$

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

$$\lambda_1 \alpha_1 v_1 + \lambda_2 \alpha_2 v_2 = 0$$

$$\lambda_1 \alpha_1 v_1 + \lambda_1 \alpha_2 v_2 = 0$$

$$0 + \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} \alpha_2 \underbrace{v_2}_{\neq 0} = 0$$

0

Can do same for λ_2 to get

$\alpha_1 = 0$.

$\therefore$ lin. indep

Induct

$$A\underline{x} = \lambda\underline{x} \Rightarrow A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda^2 x$$

$$A^n\underline{x} = A(A^{n-1}\underline{x})$$

$$= A(\lambda^{n-1}x) = \lambda^{n-1}(Ax)$$

$$= \lambda^n\underline{x}$$

$$f(A) = a_0 I + a_1 A + \dots + a_n A^n$$

$$f(\lambda) = a_0 + a_1\lambda + \dots + a_n\lambda^n$$

$$f(A)\cdot x = (a_0 I + a_1 A + \dots + a_n A^n)x$$

$$= a_0 I x + a_1 A x + \dots + a_n A^n x$$

$$= a_0 x + a_1 \lambda x + \dots + a_n \lambda^n x$$

$$= (a_0 + a_1\lambda + \dots + a_n\lambda^n)x$$

$$= f(\lambda)x$$

$$Tr(AB) = Tr(BA)$$

$$Tr(AB)$$

$$A \in \mathbb{R}^{k \times l} \qquad B \in \mathbb{R}^{l \times k}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kl} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{l1} & \cdots & b_{lk} \end{pmatrix}$$

$$BA = \begin{pmatrix} a_{i1} b_{1i} & & & \\ & a_{i2} b_{2i} & & \\ & & \ddots & \\ & & & a_{il} b_{li} \end{pmatrix}$$

$$Tr(BA) = \sum_{j=1}^{l} \sum_{i=1}^{k} a_{ij} b_{ji}$$

$$AB = \begin{pmatrix} a_{1j} b_{j1} & & & \\ & a_{2j} b_{j2} & & \\ & & \ddots & \\ & & & a_{kj} b_{jk} \end{pmatrix}$$

$$Tr(AB) = \sum_{j=1}^{k} \sum_{i=1}^{l} a_{ji} b_{ij}$$

$\downarrow$

symmetric

$$Tr(AB) = Tr(BA)$$

$f_A(t)$ ...

(a) is monic

(b) $a_{n-1} = -Tr(A)$ t^{n-1}

$$\det(tI - A) = \begin{pmatrix} t-a_1 & & -a_{ij} \\ & t-a_{22} & \\ & & \ddots \\ -a_{ij} & & & t-a_{nn} \end{pmatrix}$$

$$= \prod_{i=1}^{n} (t - a_{ii}) = (t-a_{11})(t-a_{22})\cdots(t-a_{nn})$$

$$\rightarrow \quad -a_{11}t^{n-1} - a_{22}t^{n-1} - \cdots$$

$$\Rightarrow -\underbrace{\left(\sum_{i=1}^{n} a_{ii}\right)}_{Tr(A)} t^{n-1}$$

other terms of the determinant don't matter - will cancel out 2 t terms using cofactors so at most can have t^{n-2} factor

(c) $\det(tI - A)$

$\det(-A) = \alpha_0 = (-1)^n \det(A)$ (from class)

Show $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has no eigenbasis.

Upper triangular $\rightarrow$ $\lambda = 1$.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$y = y$ ✓

$x + y = x$

$y = 0$

$\underline{v} = (t, 0)$ $\forall t \in \mathbb{R}$ are eigenvectors

All lin dep. $\rightarrow$ cannot form a basis.

$\mathbb{R}^n$ - dim = n

adding linearly independent

$x_1, \dots, x_k,$ + x_{k+1} vectors

$\{x_1, \dots, x_n\}$ lin. indep → maximal lin independent

↓

cannot add more?

$x_i = \sum_j x_{ij} e_j$

$a = a_1 e_1 + \dots + a_n e_n$

First Miracle -

$|B| = n$

$n \times n$ triangular matrix.

n distinct eigenvalues (diagonal elements)

n lin. indep eigenvectors (HW 5)

From corollary, n lin. indep. vectors

form a basis.

↪ eigenbasis.

$$f(t) = \prod_{i=1}^{k} (t - \alpha_i) \qquad g_j(t) = \frac{f(t)}{(t-\alpha_j)}$$

Suppose lin dep.

$$g_i = \sum_{\substack{j=1 \\ j \neq i}}^{k} \mu_j g_j$$

$$f(t) = g_j \cdot (t - a_1)$$

$$= (t - \alpha_1) \sum_{j=2}^{k} \mu_j \, g_2 \longrightarrow \frac{f(t)}{t-\alpha_j}$$

$$= f(t) \sum_{\substack{j=1 \\ j \neq i}}^{k} \mu_j \left(\frac{t-\alpha_i}{t-\alpha_j} \right)$$

$$\sum \beta_i g_i(t) = 0$$

$$f(t) = \prod (t - \alpha_r)$$

$$g_i(t) = \frac{f(t)}{t - \alpha_i}$$

$t = \alpha_1$: $\beta_1 g_1(t) + \cdots + \beta_n g_n(t) = 0$

$\forall i \in [n],\ i \neq 1$

$g_i(\alpha_1) = 0$ (has $\alpha_1 - \alpha_1$)

$\beta_1 g_1(\alpha_1) = 0$

$$\beta_1 \underbrace{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \cdots (\alpha_1 - \alpha_n)}_{} = 0$$

All α_n distinct, so nonzero $\Longrightarrow$ $\beta_1 = 0$. Repeat $\forall \beta_n$

$\mathbb{R}^{\mathbb{N}}$ = space of infinite sequences

$S: \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$

$(\alpha_0, \alpha_1, \alpha_2, \ldots) \mapsto (\alpha_1, \alpha_2, \alpha_3, \ldots)$

Find all eigenvalues and corresponding eigenvectors.

$\alpha_0' = (\alpha_0', \alpha_1', \alpha_2', \ldots)$

$S\alpha_0' = (\lambda\alpha_0', \lambda\alpha_1', \lambda\alpha_2', \ldots) = (\alpha_1', \alpha_2', \alpha_3', \ldots)$

$\lambda\alpha_0' = \alpha_1' \qquad (\alpha_0', \lambda\alpha_0', \lambda^2\alpha_0', \ldots)$

$\lambda\alpha_1' = \alpha_2' \qquad (\forall\lambda)(\forall\alpha_0')$

(λ is an eigenvalue with eigenvector $(\alpha_0', \lambda\alpha_0', \lambda^2\alpha_0', \ldots)$)

Homework for wednesday

<u>Def.</u> Let G be a graph, and let $t \in \mathbb{N}$.

$f_G(t) = \#$ of legal colorings with t colors $(V \to [t])$

is called the chromatic polynomial. $\to$ (Tomorrow: prove this is polynomial)

$\boxed{\text{HW}}$ Let T be a tree with n vertices.

show that $f_T(t) = t(t- {}^{n-1}$

this is independent of the choice of tree.

$\boxed{\text{CH}}$ $f_{C_n}(t) = \ ?$

$\boxed{\text{HW}}$ Non-zero pairwise orthogonal vectors are linearly independent

orthogonal: dot product is 0.

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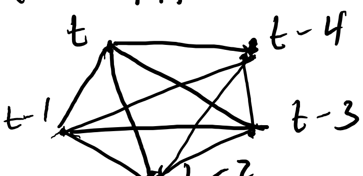
(1) $f_{\overline{K_n}}(t)$ (empty graph) - $\boxed{t^n}$ (choose 1 color from t for each of n vertices)

(2) $f_{K_n}(t)$ (complete graph)



$t(t-1)(t-2)\dots(t-4)$

$$\prod_{i=0}^{n-1}(t-i) = \frac{t!}{(t-n)!} = {}_tP_n. \quad \text{(if } t \geq n\text{, 0 otherwise)}$$

$\to k \times \ell$

$\alpha(G \square H) \geq \alpha(G)\,\alpha(H)$

Let A_G be independent set of G s.t. $|A_G| = \alpha(G)$

and A_H be independent set of H s.t. $|A_H| = \alpha(H)$.

WTS: $A_G \square A_H$ is independent set

If $g \in A_G$, $\forall h \in A_H$

$(g_1, h_1), (g_1, h_2), \ldots, (g_1, h_\ell)$ must be nonadj. since $h_1, h_2, \ldots, h_\ell$ non. adj

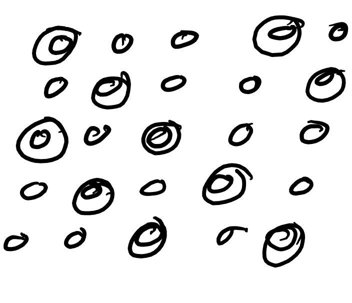
we can reason this way for $\forall g \in A_G$

$\therefore \alpha(G \square H)$ must have at least $k \times \ell$ pts. $\alpha(G)\alpha(H)$.

$\alpha(C_5 \square C_5)$

≤ 2 per column + row (can't add more b/c C_5s.)

$\alpha \leq 2 \cdot 5 = 10$



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$\alpha(G)\chi(G) \geq n$

Using colors - separate G into independent sets.

$$n = \sum_{j=1}^{\chi(G)} |V_j| \qquad |V_j| \leq \alpha(G)$$

If kl even, then $k \times l$ grid is Hamiltonian.

Exception: $1 \times n$ (n is even) $k \neq 1$, $l \neq 1$.

If k, l odd, then $k \times l$ grid not Hamiltonian

$k \times l$ grid is bipartite. Assume $k \times l$ grid is Hamiltonian. Then $k \times l$ grid has a cycle of odd length (kl is odd). Contradiction - bipartite graphs cannot have odd cycles.

$\therefore$ $k \times l$ grid is not Hamiltonian

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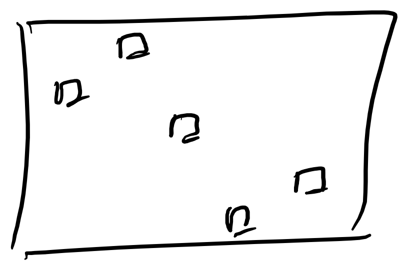
Find the smallest connected bipartite graph (equal parts) with no perfect matching.

G bipartite with equal parts.

$\det(B_G) \neq 0 \Rightarrow \exists$ a perfect matching.

$\det(B_G) \neq 0 \rightarrow$ there exists a rook arrangement / permutation s.t. product is nonzero.

This corresponds to a perfect matching (since all 1's — edges and disjoint). ☐

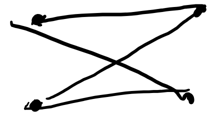


$\Rightarrow \exists$ term in sum that is nonzero.

$\rightarrow \sum \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$

Disprove $\exists$ perfect matching $\Rightarrow \det(B_G) \neq 0$.

$K_{2,2}$



$\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$.