Friday, June 30, 2017 9:34 AM

Problem Session

rank(A) = max # of lin. indep. columns.

= dim Col(A)

$\text{Col}(A) = \text{Span}\{a_1, \ldots, a_n\}$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{a_1} + x_2 \vec{a_2} + \cdots + x_n \vec{a_n}$$

$$A \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} A\vec{b_1} & A\vec{b_2} & \cdots & A\vec{b_k} \end{bmatrix}$$

If $A \in \mathbb{R}^{k \times \ell}$ and $B \in \mathbb{R}^{\ell \times m}$, then

$\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

$$\vec{b_i} = \begin{bmatrix} b_{i1} \\ \vdots \\ b_{i\ell} \end{bmatrix}$$

$A\vec{b_i} = b_{i1} \vec{a_1} + \cdots + b_{i\ell} \vec{a_\ell}$

where $\vec{a_i}$ = i^{th} column of A.

$A^* = \{a_1', \ldots, a_r'\}$
maximal lin indep. columns of A

$C^* = \{c_1', \ldots, c_s'\}$
maximal lin. indep. columns of AB

$\mathrm{rk}\ A = |A^*|$ $\qquad$ $\mathrm{rk}\ AB = |C^*|$

$(\forall i)(C_i \in \mathrm{span}\ A^*)$ $\rightarrow$ columns of AB are lin. combs. of A

(from above)

$s \leq r$

By 1st Miracle... $\Rightarrow \mathrm{rk}\ AB \leq \mathrm{rk}\ A$.

Is there a faster way to do the other half?

We know that $(AB)^T = B^T A^T$, so

$$\underset{\substack{\uparrow \\ 2^{nd}\ \text{Miracle}}}{\mathrm{rk}(AB) =} \mathrm{rk}((AB)^T) = \mathrm{rk}(B^T A^T) \underset{\substack{\uparrow \\ 1^{st}\ \text{part of proof}}}{\leq} \mathrm{rk}(B^T) \underset{\substack{\uparrow \\ 2^{nd}\ \text{Miracle}}}{=} \mathrm{rk}(B)$$

Since $\mathrm{rk}(AB) \leq \mathrm{rk}(A)$ and $\mathrm{rk}(AB) \leq \mathrm{rk}(B)$,

$\mathrm{rk}(AB) \leq \min\{\mathrm{rk}(A), \mathrm{rk}(B)\}$. $\square$

Find $A \in M_2(\mathbb{R})$ s.t. $A \neq \underline{0}$ but $A^2 = \underline{0}$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If $A \in \mathbb{R}^{\ell \times k}$ and $B \in M_k(\mathbb{R})$ is nonsingular,

$\Downarrow$

$(\det(B) \neq 0)$

then $\operatorname{rank}(AB) = \operatorname{rank}(A)$.

$\left(\begin{matrix} \text{rows} \\ \text{columns} \end{matrix} \right\}$ of B lin. indep.)

$AB \in \mathbb{R}^{\ell \times k}$

$$\left.\begin{matrix} \dim \{A^T x : x \in \mathbb{R}^\ell\} \\ \dim \{B^T A^T x : x \in \mathbb{R}^\ell\} \end{matrix}\right\} \boxed{rk(B^T A^T) = rk(A^T)}.$$

$\Uparrow$

$\operatorname{col-sp}(A^T) \longrightarrow \operatorname{col-sp}(B^T A^T)$ $\nearrow$ isomorphisms

$y \longmapsto B^T y$ (WTS: bij.)

surjectivity: inverse operations (take transpose again)

injectivity: $B^T x = B^T y$

$B^T(x - y) = 0$

$x - y = 0 \Rightarrow x = y$

Another approach:

we know $\text{rank}(AB) \le \text{rank}(A)$ from previous problem

exists b/c B nonsingular

$$A = (AB)B^{-1}$$

$$\text{rank}(A) \le \min\{\text{rank}(AB), \text{rank}(B^{-1})\}$$

$$\therefore \text{rank}(A) \le \text{rank}(AB) \quad \text{and}$$

$$\text{rank}(A) = \text{rank}(AB). \qquad \square$$

Let $(v_1, \dots, v_k) \in \mathbb{R}^n$. Apply elementary operations to these to get $(v_1', \dots, v_k') \in \mathbb{R}^n$.

Then $\dim \text{span}(v_1, \dots, v_k) = \dim \text{span}(v_1', \dots, v_k')$

Elementary operation: (i, j, λ) means

$$v_\ell' = \begin{cases} v_\ell & \ell \ne i \\ v_i - \lambda v_j & \ell = i. \end{cases}$$

Proof Pick a basis $\overset{\vee}{W}$ from among $v_1, \dots, v_k$: for span $(v_1, \dots, v_k)$

$W = w_1, w_2, \dots, w_n$.

Check case where $v_j \in W$. (If $v_i \notin W$, will not change span b/c not a basis vector.)

$$\alpha_1 w_1 + \underset{\text{w/o } v_i}{\dots} + \alpha_n w_n = v_i - \lambda v_j$$

$$\alpha_1 w_1 + \underset{\text{w/o } v_i}{\dots} + \alpha_n w_n + \lambda v_j = v_i$$

If $w_1, \dots, w_n$ is LI $\to$ basis and $w_1, \underset{\text{w/o } v_i}{\dots}, w_n, v_i - \lambda v_j$ is LI $\to$ basis .. ?

X

Proof. $w \in \text{Span}(v_1', \dots, v_k')$

$w = \alpha_1 v_1 + \dots + \alpha_i(v_i - \lambda v_j) + \dots \alpha_k v_k$

(lin comb. of normal v_j's $\Rightarrow w \in \text{Span}(v_1, \dots, v_k)$)

Inverse of elementary operation is elementary operation ⇒ make symmetric argument in other direction w/ inverse to show

$w \in \text{Span}(v_1', \dots, v_k') \Rightarrow w \in \text{Span}(v_1, \dots, v_k)$.

Thus, $\text{Span}(v_1, \dots, v_k) = \text{Span}(v_1', \dots, v_k')$

and it follows that

$\dim \text{Span}(v_1, \dots, v_k) = \dim \text{Span}(v_1', \dots, v_k')$. $\square$

$\therefore$ Elementary column operations do not change the column rank.

Elementary <u>row</u> operations do not change the column rank.

(Prove w/o 2nd Miracle.)

If the column rank of A is ℓ, then

$\exists\ \ell$ linearly independent columns

$a_1, \ldots, a_\ell$.

Apply some row operation,

It would be nice if the <u>same</u> columns

were linearly independent post-transformation.

WTS: $a_1', \ldots, a_\ell'$ are linearly independent.

$$a_1' = \begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} - \lambda a_{j1} \\ \vdots \\ a_{n1} \end{pmatrix}$$

$\alpha_1 a_1' + \cdots + \alpha_\ell a_\ell' = 0$,

$$\alpha_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{i1} - \lambda a_{j1} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + \alpha_\ell \begin{pmatrix} a_{1\ell} \\ \vdots \\ a_{i\ell} - \lambda a_{j\ell} \\ \vdots \\ a_{n\ell} \end{pmatrix} = 0$$

Assume linear <u>dependence</u> at least one of α_i nonzero.

Take i^{th} and j^{th} rows.

$$\alpha_1 (a_{i1} - \lambda a_{j1}) + \cdots + \alpha_\ell (a_{i\ell} - \lambda a_{j\ell}) = 0$$

$$\lambda\big(\alpha_1 (a_{j1}) + \cdots + \alpha_\ell (a_{j\ell}) = 0\big)$$

$$\alpha_1 (a_{i1}) + \cdots + \alpha_\ell (a_{i\ell}) = 0$$

The same set of coefficients makes all the other rows 0 as well, so

$$\alpha_1 \vec{a_1} + \alpha_2 \vec{a_2} + \cdots + \alpha_\ell \vec{a_\ell} = 0$$

coefficients nontrivial $\Rightarrow$ original columns lin. dep.

Contrapositive $\Rightarrow$ lin indep $a_1, \ldots, a_\ell \Rightarrow$ lin indep. $a_1', \ldots, a_\ell'$.

Row ops. don't decrease column rank.
(cannot increase either $\Rightarrow$ inverse is a row op. and would decrease.) $\Rightarrow$ don't change them □

If $\varphi : V \to W$ is a linear map, then

$\varphi(0_V) = 0_W$.

Contra $\varphi(0_V) \neq 0_W \Rightarrow \varphi$ nonlinear.

$\exists\, w \in W$ s.t. $w \neq 0$ and $\varphi(0_V) = w$

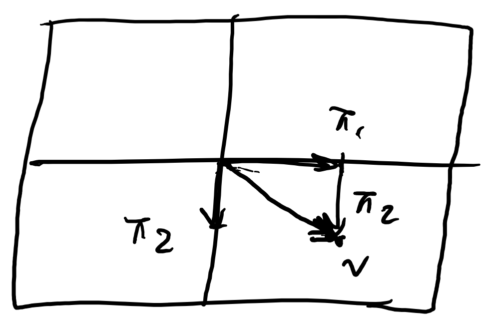
$2\varphi(0_V) = 2w \neq w$

$2\varphi(0_V) \neq \varphi(2 \cdot 0_V) = w$

$\therefore \varphi$ not linear $\square$

If $\varphi, \psi \in \mathrm{Hom}(V, W)$, then $\varphi + \psi \in \mathrm{Hom}(V, W)$.

Example. G_2



π_1 : proj. onto x-axis

π_2 : proj. onto y-axis.

$\pi_1 + \pi_2$ = identity map.

In higher dimensions $\Rightarrow$ proj. onto xy-plane

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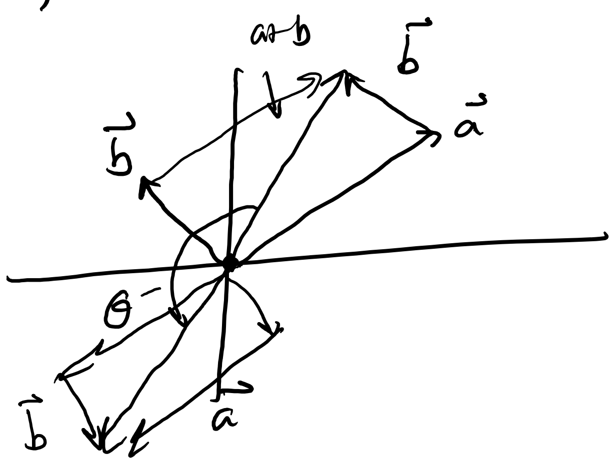
rot_θ = rotation by θ (counterclockwise) about origin.

When is rot_θ not linear?

↳ not about origin.

Why is rotation about origin linear?



If rotate parallelogram, get a parallelogram.

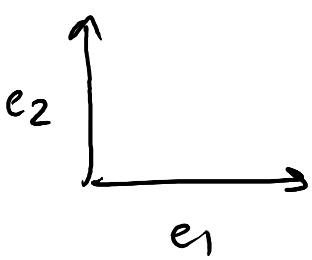
Now ignore $\theta = k\pi$ w/ $k \in \mathbb{Z}$.

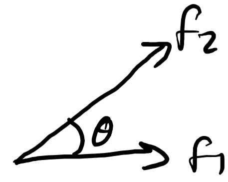
$\{f_1, f_2\}$

$\{e_1, e_2\}$

$f_1 = e_1$

$f_2 = rot_\theta(f_1)$





$$[rot_\theta]_{\underline{e}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad [rot_\theta]_{\underline{f}} = ?$$

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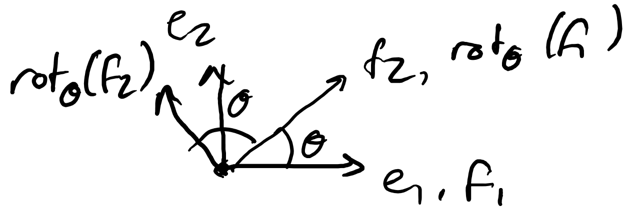


$$\mathrm{rot}_\theta(f_1) = \mathrm{rot}_\theta(e_1) = \underline{\underline{f_2}}$$

$$\mathrm{rot}_\theta(f_2) = \mathrm{rot}_\theta(\mathrm{rot}_\theta(e_1)) = \mathrm{rot}_{2\theta}(e_1)$$

$$f_2 \mapsto \cos 2\theta \, e_1 + \sin 2\theta \, e_2$$

$$= \cos 2\theta \, f_1$$

want to find e_2 in terms of f_1 and f_2.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(90-\theta) & -\sin(90-\theta) \\ \sin(90-\theta) & \cos(90-\theta) \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

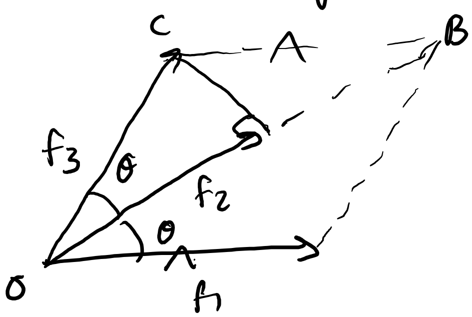
$$f_2 \mapsto \cos(2\theta) f_1 + \sin(2\theta) M f_2$$

$$\begin{bmatrix} 0 & \cos 2\theta \\ 1 & \sin 2\theta \, \underline{M} \end{bmatrix}$$

$\hookrightarrow$?

X

A more geometric approach . .



$f_2 = \mathrm{rot}_\theta(f_1)$

$f_3 = \mathrm{rot}_\theta(f_2)$

$f_3 = \vec{OB} + \vec{BC}$

$\vec{OB} = 2\cos\theta\, f_2$

$\vec{BC} = -f_1$

$$\begin{bmatrix} 0 & -1 \\ 1 & 2\cos\theta \end{bmatrix}$$

<u>Lecture</u>

<u>Coordinatization</u>

V basis $(e_1, \dots, e_n) = \underline{e}$

$v \in V$ $v = \sum \alpha_i e_i$

$$[v]_{\underline{e}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n.$$

$v \longmapsto [v]_{\underline{e}}$

$V \longmapsto \mathbb{R}^n$

<u>linear bijection</u>

$\parallel$

isomorphism

$\therefore$ If $\dim V = n$ then $V \cong \mathbb{R}^n$.

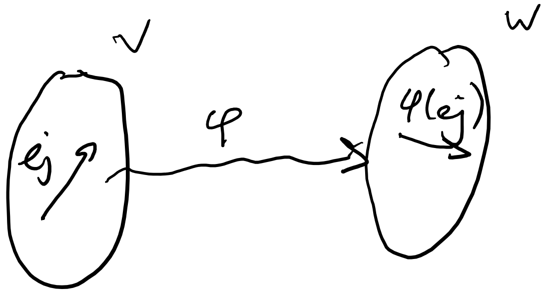
$\varphi : V \to W$

$\underline{e} = (e_1, \dots, e_n)$ and $\underline{f} = (f_1, \dots, f_k)$

$[\varphi]_{\underline{e}, \underline{f}} \in \mathbb{R}^{k \times n}$

j^{th} column of $[\varphi]_{\underline{e}, \underline{f}}$ is $[\varphi(\underline{e_j})]_{\underline{f}}$.

image of basis vectors from first space.



$\varphi \longmapsto [\varphi]_{\underline{e}, \underline{f}} \in \mathbb{R}^{k \times n}$

$\mathrm{Hom}(V, W) \longmapsto \mathbb{R}^{k \times n}$

$\mathbb{R}^{k \times n}$ forms a vector space $\to$ a matrix, add + scaler multiply.

Def $(\varphi + \psi)(v) = \varphi(v) + \psi(v)$

$(\forall v \in V)$

$\lambda(\varphi(v)) = \varphi(\lambda v)$

Forms a vector space as well.

Claim: $\mathrm{Hom}(V, W) \mapsto \mathbb{R}^{k \times n}$ is an isomorphism of vector spaces by

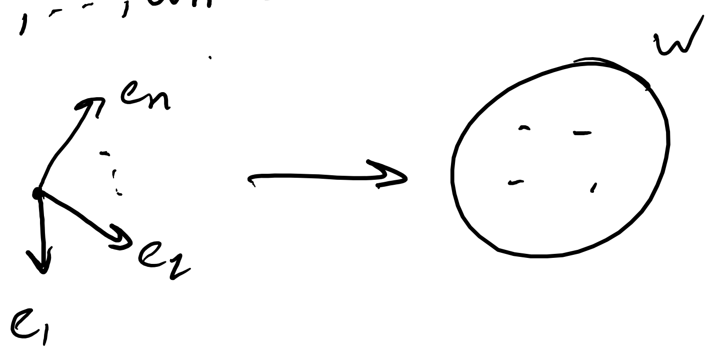
(1) observing that this correspondence is linear

(2) bijection.

<u>Thm</u> (from yesterday.)

($\forall$ basis $e_1, \ldots, e_n$ of V)

$(\forall w_1, \ldots, w_n \in W)(\exists! \varphi : V \to W)(\forall i)(\varphi(e_i) = w_i)$



extends uniquely to a linear map.

$\Rightarrow$ these basis vector mappings form bij. w/ linear maps.

Uniqueness: given the images of $\underline{e_1}, \dots, \underline{e_n}$, we know the image of any $v \in V$.

if $v \in V$, $v = \sum \alpha_i e_i$ (from basis)

If $(\exists \varphi)(\forall i)(\varphi(e_i) = w_i)$,

$$\underline{\varphi(v)} = \varphi\left(\sum \alpha_i e_i\right) = \sum \alpha_i \varphi(e_i) = \underline{\sum \alpha_i w_i}.$$

lin. map. properties.

Existence

Define $\varphi : V \to W$ by setting $(\forall v \in V)(\text{if } v = \sum_i \alpha_i e_i \text{ then } \varphi(v) := \sum \alpha_i w_i)$.

NTS: this $V \to W$ map is linear. DO.

$\therefore \exists$ bijection

$$\mathrm{Hom}(V, W) \to W^{\{e_1, \dots, e_n\}}$$

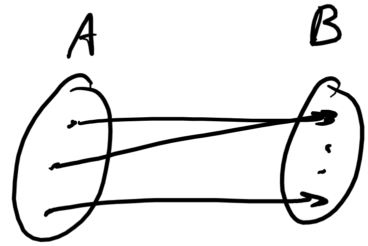
Notation: If A, B are sets then

$$B^A = \{f : A \to B\}.$$

If A, B finite, then

$$|B^A| = |B|^{|A|}.$$

(This justifies the notation)



for each pt in A,

$|B|$ choices ...

$$\underbrace{|B| \cdot |B| \cdot \cdots \cdot |B|}_{|A| \text{ times}} = |B|^{|A|}$$

$\mathbb{R}^{\mathbb{R}}$: $\mathbb{R} \to \mathbb{R}$

$\mathbb{R}^{\mathbb{N}}$: sequences : $\mathbb{N} \to \mathbb{R}$ (function)

$$\mathbb{R}^n := \mathbb{R}^{[n]} \ni \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a(1) \\ \vdots \\ a(n) \end{bmatrix}$$

Bijection: $\operatorname{Hom}(V, W) \to W^{\{e_1, \ldots, e_n\}} \to \mathbb{R}^{k \times n}$

$\varphi \mapsto [\varphi]$ $\qquad [w_1]_{\underline{f}} \ldots$

$w_1 \; w_2 \; \ldots \; w_n$

$\varphi \mapsto$

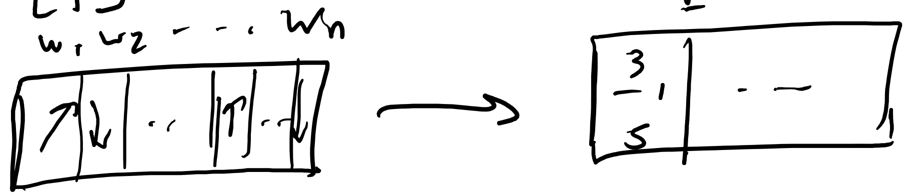
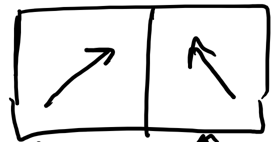
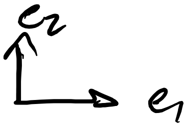


image of $e_1, \ldots, e_n$ in columns $\to$ vectors in W.

$\underline{f}$ (coordinatization)

Ex. $\mathrm{rot}_\theta \longmapsto$  $\longmapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\mathrm{rot}_\theta(e_1)$ $\mathrm{rot}(e_2)$ coordinatization.



(vectors in columns)

$k = \dim W$

$\therefore \ \mathrm{Hom}(V, W) \cong \mathbb{R}^{k \times n}$ $\quad n = \dim V$.

Not canonical ... depend on choice of basis.

what happens if you <u>change the basis</u>?

$\underline{e} = (\underline{e}_1, \ldots, \underline{e}_n)$ $\quad \underline{e}' = (\underline{e}_1', \ldots, \underline{e}_n')$

"old" "new"

$[v]_{old}$ -- ? -- $[v]_{new}$

By Thm. $(\exists!\ \sigma : V \to V)(\forall i)(\sigma(e_i) = e_i')$.

\sigma

Now let $[v]_{old} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ and $[v]_{new} = \begin{bmatrix} \alpha_1' \\ \vdots \\ \alpha_n' \end{bmatrix}$.

$$v = \sum \alpha_i e_i$$

$$\sigma(v) = \sum \alpha_i \sigma(e_i) = \sum \alpha_i e_i'$$

$$\underline{x} = [v]_{old} = [\sigma(v)]_{new} = [\sigma]_{\underline{new}} [v]_{\underline{new}}$$

or $\underline{x}' = [v]_{new} = [\sigma]_{new}^{-1} [v]_{old}$

$$S = [\underline{e}']_{\underline{e}} = [\sigma]_{old}$$

$$= [[e_1']_{old}, \ldots, [e_n']_{old}]$$

"basis change matrix"

DO $[\sigma]_{new} = [\sigma]_{old}$.

$\boxed{\underline{x}' = S^{-1}\underline{x}}$ where S is the basis change matrix.

Ex. What if $e_i' = 2e_j$?

$$v = \sum \alpha_j e_i$$
$$= \sum \alpha_i' e_i'$$

$e_i' = 2e_j$ so $\alpha_i' = \frac{\alpha_j}{2}$.

$$\overset{n}{\varphi: V} \rightarrow \overset{k}{W}.$$

$\underline{e}$, $\underline{f}$
$\underline{e}'$ $\underline{f}'$

Suppose $A = [\varphi]_{old} = [\varphi]_{\underline{e},\underline{f}}$.

$A' = [\varphi]_{new} = [\varphi]_{\underline{e}',\underline{f}'}$

$\sigma : \underline{e} \rightarrow \underline{e}'$ $\tau : \underline{f} \rightarrow \underline{f}'$

\tau

$[\sigma] : S$ $[\tau] : T$.

$$\boxed{\begin{array}{l}\underline{\text{Thm.}}\\ A' = T^{-1} A S.\end{array}}$$

$k \times k$ $k \times n$ $n \times n$

DO

Basis change for lin. transformations:

$$A' = S^{-1} A S$$

$\underline{\text{Def}}$ $A, B \in M_n(\mathbb{R})$.
A is $\underline{\text{similar}}$ to B $(A \sim B)$ if $\exists S, S^{-1} \in M_n(\mathbb{R})$ s.t.
$\underline{B = S^{-1} A S}$.

$\boxed{\text{HW}}$ If $A \sim B$, then $f_A = f_B$.

(characteristic polynomials are equal)

(Similar matrices have the same eigenvalues.)

<u>Def</u>. A is <u>diagonalizable</u> if

$$A \sim \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

$\boxed{\text{HW}}$ Find a non-diagonalizable 2×2 matrix that <u>has</u> a real eigenvalue.

$\boxed{\text{HW}}$ Find a 2×2 real matrix with no <u>real</u> eigenvalues.

$\boxed{\text{HW}}$ Find the complex roots of the characteristic polynomial of the rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

$\boxed{\text{HW}}$ Prove: A is diagonalizable $\Leftrightarrow$
A has an eigenbasis.

$\boxed{\text{HW}}$ Given $a, b, c \in \mathbb{R}$, prove $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is diagonalizable.

Reminder:
Problems from previous lectures due next Wed:
Problems from previous lectures - see website
- Chromatic polynomial problem
- If $A \in \mathbb{R}^{k \times l}$, then $rk(A^T A) = rk(A)$.
- If $G \not\supset K_3$ (triangle-free), then $\chi(G) = O(\sqrt{n})$, (big-Oh)
  i.e. $(\exists c)(\forall$ sufficiently large $n \longrightarrow \chi(G) \leq c\sqrt{n})$
  estimate implied constant.
- Suppose we have
  $A_1, \ldots, A_m, B_1, \ldots, B_m \in M_n(\mathbb{R})$
  s.t. $A_i B_j = B_j A_i \Leftrightarrow i \neq j$.

  Prove: $m \leq n^2$.
  (Hint: Show $A_1, \ldots, A_m$ are linearly independent)