

Week 5 - Day 5.

$$f(t) = t^4 + at^3 + bt^2 + ct - 15 \quad a, b, c \in \mathbb{Z}$$

$$\gcd(r, s) = 1, \quad f(\frac{r}{s}) = 0 \Rightarrow s=1, \quad r \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$$

Rational roots of integral polynomials

(D)

Suppose we know  $s=1 : f(r) = 0 \quad r \in \mathbb{Z}$

$$r^4 + ar^3 + br^2 + cr - 15 = 0$$

$$r(r^3 + ar^2 + br + c) = 15$$

Spectral Thm  $A \in M_n(\mathbb{R})$ ,  $A = A^T \Rightarrow \exists$  ON eigenbasis

(D) Actually, every ON list of eigenvectors can be extended to an eigenbasis.

SVD Thm (Singular Value Decomposition)

$\mathbb{R} \ni \varphi : V \rightarrow W$  Euclidean spaces  $\operatorname{rk}(\varphi) = r$

$\Rightarrow \exists \sigma_1, \sigma_2, \dots, \sigma_r \geq 0$

(called singular values of  $\varphi$ )

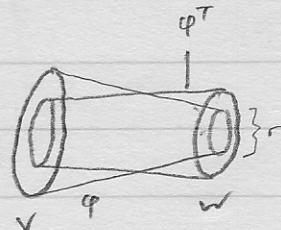
$\exists$  ONB  $e_1, \dots, e_r \in V$ ,  $f_1, \dots, f_r \in W$

s.t.  $(\forall i \leq r)(\varphi e_i = \sigma_i f_i)$

$(\forall i \leq r)(\varphi^T f_i = \sigma_i e_i)$

$(\forall i > r)(\varphi e_i = 0)$

$(\forall i > r)(\varphi^T f_i = 0)$



$\operatorname{rk}(\varphi) \stackrel{\text{def.}}{=} \dim(\operatorname{im}(\varphi))$

$\varphi^T : W \rightarrow V$

Equivalent:

$A \in \mathbb{R}^{k \times l}$

$\Rightarrow \exists S, T \in O(n)$

(orthogonal:  $S^{-1} = S^T$  and  $T^{-1} = T^T$ )

i.e.  $[\varphi]_{e,f} = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & 0 \\ & & \sigma_r & 0 \\ 0 & 0 & \ddots & 0 \end{bmatrix}$

s.t.  $S^{-1}AT = (*)$

Equiv statement of Spectral Thm

If  $A \in M_n(\mathbb{R})$ ,  $A = A^T$  then ( $\exists S \in O(n)$ ) ( $S^{-1}AS = \text{diag} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ )

$\varphi^T \varphi : V \rightarrow V$

symmetric:  $(\varphi^T \varphi)^T = \varphi^T \varphi^{TT} = \varphi^T \varphi$

pos. semi definite:  $\langle x, \varphi^T \varphi x \rangle = \langle \varphi x, \varphi x \rangle = \|\varphi x\| \geq 0$

nullity = multiplicity of 0

$\therefore \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0 = \lambda_{r+1} = \dots = \lambda_n$

Choose

eigenvalues of  $\varphi^T \varphi$ .

$$\sigma_i := \sqrt{\lambda_i} \quad i \in [r]$$

Proof of SVT  $\rightarrow$  This is the only possibility.

Let  $e$  be an ONeigenspace of  $\varphi^T \varphi : V \rightarrow V$

We must choose  $f_i$  s.t.  $\underbrace{\varphi e_i}_{\text{known}} = \sigma_i f_i \checkmark$

for  $i \leq r$   $\uparrow$   $f_i := \frac{1}{\sigma_i} \varphi e_i \quad i \leq r$

$$\text{Now } \varphi^T f_i = \frac{1}{\sigma_i} \varphi^T \varphi e_i = \frac{1}{\sigma_i} \underbrace{\lambda_i e_i}_{\sigma_i^2 e_i} = \frac{\sigma_i^2}{\sigma_i} e_i = \sigma_i e_i \checkmark$$

Need to show:

$$\forall i, j \leq r \quad \langle f_i, f_j \rangle = \delta_{ij}.$$

$$\begin{aligned} \left\langle \frac{1}{\sigma_i} \varphi e_i, \frac{1}{\sigma_j} \varphi e_j \right\rangle &= \frac{1}{\sigma_i \sigma_j} \langle \varphi e_i, \varphi e_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle e_i, \varphi^T \varphi e_j \rangle \\ &= \frac{1}{\sigma_i \sigma_j} \langle e_i, \lambda_j e_j \rangle = \frac{\lambda_j}{\sigma_i \sigma_j} \underbrace{\langle e_i, e_j \rangle}_{\delta_{ij}} \end{aligned}$$

$$\text{so } \langle f_i, f_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \frac{\lambda_i}{\sigma_i^2} = 1 & \text{if } i = j \end{cases} \quad \delta_{ij}$$

extend  $f_1, \dots, f_r$  to ONB of  $V$ .

$i > r$  Claim:  $\varphi e_i = 0$  NTS:  $\|\varphi e_i\| = 0$

(DO)  $\varphi^T f_i = 0 \quad \forall i > r$ .

$$\begin{aligned} \langle \varphi e_i, \varphi e_i \rangle &= \langle e_i, \varphi^T \varphi e_i \rangle \\ &= \langle e_i, \lambda_i e_i \rangle = 0. \end{aligned}$$

Thus we have verified the conditions for singular value decomposition.  $\square$

Operator norm

$\varphi: V \rightarrow W$  Euclidean spaces

$$\|\varphi\| = \max_{\substack{x \in V \\ x \neq 0}} \frac{\|\varphi x\|_W}{\|x\|_V}$$

(DO) Prove max is attained.

(DO) If  $\varphi: V \rightarrow V$ ,  $\varphi^T = \varphi$ ,  $\underbrace{\lambda_1, \lambda_2, \dots, \lambda_n}_{\text{eigenvalues}}$

$$A \in \mathbb{R}^{k \times k}$$

$$\|A\| = \max_{x \in \mathbb{R}^k} \frac{\|Ax\|_V}{\|x\|_V}$$

Then  $\|\varphi\| = \max |\lambda_i|$ .

$$\begin{aligned} \text{General case: } \|\varphi\|^2 &= \max \frac{\|\varphi x\|^2}{\|x\|^2} = \max \frac{\langle \varphi x, \varphi x \rangle}{\|x\|^2} \\ &= \max \frac{\langle x, \varphi^T \varphi x \rangle}{\|x\|^2} = \max_{\substack{x \in \mathbb{R}^k \\ x \neq 0}} \varphi^T \varphi(x) = \max |\lambda_i(\varphi^T \varphi)| \\ &= \text{some } "k \text{ abs. val.}" \end{aligned}$$

Thm.  $\|\varphi\| = \sqrt{\max \lambda_i \text{ of } \varphi^T \varphi}$ .

$\|A\| = \sqrt{\max \lambda_i \text{ of } A^T A}$ .

Thm. (Low-rank approximation.) Fix threshold  $t$ .

Given  $A \in \mathbb{R}^{k \times l}$ , we wish to find  $B \in \mathbb{R}^{k \times l}$  s.t.

(i)  $\text{rk}(B) \leq t$

(ii)  $\|A - B\|$  is minimized.

Then  $B = S \otimes T^{-1}$  (see right).

$$S^T A T = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & 0 \\ & & \ddots & \vdots \\ 0 & & & \sigma_t \end{pmatrix} \quad (\text{TEO}(l))$$

$$\text{t-truncation} = \begin{pmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & 0 \\ & & \ddots & \vdots \\ 0 & & & \sigma_t \end{pmatrix} \quad (\otimes)$$

$$\varphi: V \rightarrow W$$

$\ell \quad k$

$\varphi: e_i \mapsto \sigma_i f_i \quad i \leq r$  (assume  $t < r$ )  
 $\varphi: e_i \mapsto \sigma_i f_i \quad i \leq t$  (if not - done.)  
 $\varphi: e_i \mapsto 0 \quad i > t$

$\varphi_t: t\text{-truncation of } \varphi$

$$\varphi_t: e_i \mapsto \sigma_i f_i \quad i \leq t$$

Suppose  $\psi: V \rightarrow W$ , rk  $\psi \leq t$ .

$$\underline{\text{Claim}} \quad \|\varphi - \psi\| \geq \|\varphi - \varphi_t\| = \sigma_{t+1}$$

$$U_j := \text{span}(e_1, \dots, e_j)$$

$$\dim \ker \psi \geq \underbrace{\dim}_{\ell} V - t$$

(by rank-nullity)

$$C = \begin{pmatrix} 0 & & & \\ & \ddots & & 0 \\ & & 0 & \\ & & & \ddots & 0 \\ & & & & 0 & \end{pmatrix}$$

$$C^T C = \begin{pmatrix} 0 & & & \\ & \ddots & & 0 \\ & & 0 & \\ & & & \ddots & 0 \\ & & & & 0 & \end{pmatrix}$$

$$\therefore \ker \psi \cap U_{t+1} \neq \{0\}.$$

$$\lambda_{\max}(C^T C) = \sigma_{t+1}^2$$

$$\geq \ell - t + t + 1 > \ell, \text{ so must have nontrivial intersection.}$$

Take  $v \neq 0$  s.t.  $v \in \ker \psi \cap U_{t+1}$ .

$$\text{NTS: } \|\varphi - \psi\| \geq \sigma_{t+1} \quad \psi: V \rightarrow W, \text{ rk } \psi \leq t$$

$$\|\varphi - \psi\| = \max_{\substack{x \in V \\ x \neq 0}} \frac{\|\varphi(x) - \psi(x)\|}{\|x\|} \geq \frac{\|\varphi(v) - \psi(v)\|}{\|v\|} = \frac{\|\varphi v\|}{\|v\|} \geq \min_{y \in U_{t+1}} \frac{\|\varphi y\|}{\|y\|}$$

$\psi v = 0 \text{ b/c } v \in \ker \psi. \quad \uparrow \quad v \in U_{t+1}$

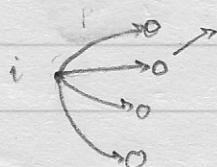
$$\|\varphi - \psi\|^2 \geq \min_{y \in U_{t+1}} \frac{\|\varphi y\|^2}{\|y\|^2} = \min_{y \in U_{t+1}} \frac{\langle \varphi y, \varphi y \rangle}{\|y\|^2} = \min_{y \in U_{t+1}} \frac{\langle y, \varphi^T \varphi y \rangle}{\|y\|^2} = \min_{y \in U_{t+1}} R_{\varphi^T \varphi}(y)$$

= min  $\lambda$  of

$$(R_{\varphi^T \varphi}|_{U_{t+1}})$$

$$= \lambda_{t+1} = \sigma_{t+1}^2. \quad \square$$

Markov Chain: memoryless random walk.



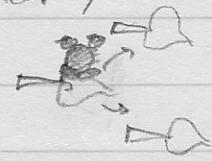
n "states"

$P_{ij} = \text{prob. of transition from state } i \text{ to state } j$ .

$X_t$  = location of marker at time  $t$  ("leaf")

$P_{ij} = \text{prob}(X_{t+1} = j | X_t = i)$

given (conditional prob.)



transition matrix:  $T = (P_{ij})_{n \times n}$ .

$P_{ij} \geq 0$  and  $\sum_{j=1}^n P_{ij} = 1$  (row sums = 1)  $\leftarrow$  stochastic matrix.

Evolution of Markov Chain.

$g_t = (g_{t1}, \dots, g_{tn})$  is prob. distribution.

meaning:  $P(X_t = i) = g_{ti}$

$g_{ti} \geq 0 \quad \forall i$

$\sum_{i=1}^n g_{ti} = 1$

Def. A probability distribution  $g$  is a stationary distribution if it does not change from time  $t$  to  $t+1$ .

Then  $g_{t+1} = g_t \cdot T$  (note  $g_t$  are row vectors)

(DO)

Evolution Equation of MC:  $g_t = g_0 \cdot T^t$

stationary if  $g = gT$ ; i.e.  $g$  is a left eigenvector to eigenvalue 1. (nontrivial to show existence.)

Find right eigenvector to eigenvalue 1.

$$\Rightarrow T \begin{pmatrix} 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \end{pmatrix}$$

(DO) Right eigenvalues = left eigenvalues. (same char poly between  $T$  and  $T^T$ )

(DO)  $T$  and  $T^T$  have same geom. mult. of eigenvalues.

Perron - Frobenius Thm. for nonnegative matrices: (highly nontrivial)

If  $A = (a_{ij}) \in M_n(\mathbb{R})$ ,  $a_{ij} \geq 0$ , then  $\exists$  non-negative eigenvector.

(corresponding eigenvalue  $\geq 0$ )

Associated digraph (directed graph):

$i \rightarrow j$  if  $a_{ij} > 0$   
(edge)

If MC and associated digraph is strongly connected (can get anywhere on graph),  
then there is unique eigenvalue  $\geq 0$ .

Def A Markov Chain is ergodic if assoc. digraph:

(1) strongly connected

(2) aperiodic :  $\text{gcd}(\text{lengths of all dir. cycles}) = 1$

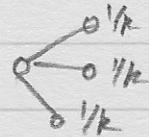
Thm. In this case, stationary distribution  $\pi$  is unique and

$$(\forall g_0) (\lim_{t \rightarrow \infty} g_t = \pi).$$

Naive random walk on a  $k$ -regular graph -

$$T = \frac{1}{k} A \quad \text{adj. matrix}$$

$$\hookrightarrow \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$



graph connected:  $\Leftrightarrow \lambda_1 < 1$

$$\lambda_1 = 1$$

$$\sum \lambda_i = 0$$

If G bipartite,

$$\lambda_n = -\lambda_1$$

$$\lambda_{n-1} = -\lambda_2$$

$\vdots$

not bipartite  $\Rightarrow \lambda_n \geq -1$ .

Thm. (Convergence for naive random walk on  $k$ -reg. graph)

$$\lambda := \max \{|\lambda_2|, \dots, |\lambda_n|\}$$

$$|P_{ij}^{(t)} - \frac{1}{n}| \leq \lambda^t \quad P_{ij}^{(t)} = (T^t)_{ij} \quad (\lambda_1 = 1)$$

$t$ -step transition probability.

(DO) If  $T$  symm. Then  
 $(\frac{1}{n}, \dots, \frac{1}{n})$  is stationary.

(convergence rate estimated by eigenvalue gap  $1 - \lambda_2$ )

(DO)  $A = (a_{ij}) \Rightarrow \|A\| \geq |a_{ij}|$

stronger conclusion:  $\|T^t - \frac{1}{n}I\| \leq \lambda^t$   
 $\underbrace{\text{symmetric matrix.}}$

Take ON eigenbasis of  $T$ :  $e_1, \dots, e_n : e_1 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

Claim:  $e_1, \dots, e_n$  is an eigenbasis

$$Te_1 = \lambda_1 e_1, \quad \lambda_1 = 1.$$

of  $B = T^t - \frac{1}{n}I$ .

extend to  
 $\underbrace{\text{one eigenvector}}_{\text{one eigenvector}} \rightarrow Te_i = \lambda_i e_i$

$$|\lambda_i| \leq 2$$

$$T^t e_i = \lambda_i^t e_i$$

$$\frac{1}{n} I e_i = e_i$$

$$\frac{1}{n} I e_i = 0 \quad \forall i \geq 2.$$

$$\left[ \begin{array}{c|c} e_1^T & \vdots \\ \hline \dots & \dots \\ e_i^T & \vdots \\ \hline e_n^T & \vdots \end{array} \right] \left[ \begin{array}{c} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_n \end{array} \right] = 0$$

$\therefore$  eigenbasis.

$$Be_i = \underbrace{T^t e_i}_{e_i} - \underbrace{\frac{1}{n} I e_i}_{e_i} = 0$$

eigenvalues of  $B$  are  
 $0, \lambda_2^t, \lambda_3^t, \dots, \lambda_n^t$

$\forall i \geq 2$

so max is  $\lambda_n^t$

$$Be_i = \underbrace{T^t e_i}_{\lambda_i^t e_i} - \underbrace{\frac{1}{n} I e_i}_0 = \lambda_i^t e_i \quad (\forall i \geq 2) \quad (1|\lambda_i^t| \leq \lambda^t). \quad \square$$

Thm. If  $G$  is  $k$ -regular graph of girth  $\geq 5$  and  $n = k^2 + 1$ ,  
then  $k \in \{1, 2, 3, 7, 57\}$ .

$$n \geq k^2 + 1$$

(previous Hw)

$G$  graph  $A$  adj matrix

$$A = (a_{ij}) \quad a_{ij} = \begin{cases} 1 & i \sim j \\ 0 & i \not\sim j \end{cases} \quad (\text{note: } a_{ii} = 0) \quad \begin{array}{ll} k=1 & 0-0 \\ k=2 & \text{adj} \\ k=3 & \text{Peterson's graph} \\ k=7 & \text{Hoffmann-Singleton graph} \\ k=57 & ? \end{array}$$

$$A^2 = (b_{ij}) \quad b_{ij} = \sum_{k=1}^n a_{ik} a_{kj}$$

$$= |\{k \mid a_{ik}=1 \text{ and } a_{kj}=1\}|$$

$$= |\{(k \mid k \sim i, k \sim j)\}|$$

$$= \# \text{ common neighbors of } i \text{ and } j.$$

In particular,  $b_{ii} = \deg i$ .

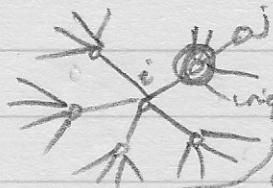
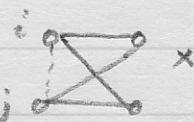
$$b_{ii} = k.$$

If  $i \sim j$  then  $b_{ij} = 0$  (bc then would create a 3-cycle)

If  $i \not\sim j$  then  $b_{ij} \leq 1$ .

$\curvearrowleft$  (2 would create 4-cycle)

Claim If  $n = k^2 + 1$  then  $b_{ij} = 1$  if  $i \not\sim j$ .



$$\text{degree common neighbor. } 1 + k + k(k-1) = k^2 + 1$$

$\curvearrowleft$  (k-1 verts.) If  $n = k^2 + 1$  then all vertices are within dist. 2 of  $i$ . if  $i \not\sim j$ .

$\bar{A} = \bar{A}_0$  : adjacency matrix of complement  $\bar{A}_0$ .

$$A + A^2 = \begin{pmatrix} k & 1 \\ 1 & k \end{pmatrix} = kI + (J-I) = (k-1)I + J$$

$$a_{ij} + b_{ij}$$

$$A^2 + A - (k-1)I = J$$

$$\begin{aligned} a_{ij} = 1 &\Rightarrow b_{ij} = 0 \\ a_{ij} = 0 &\Rightarrow b_{ij} = 1 \end{aligned} \quad \left\{ \begin{array}{l} a_{ij} + b_{ij} = 1 \end{array} \right.$$

Take ONEigenbasis of  $A$ , starting with  $e_1 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$e_1, \dots, e_n \quad Ae_i = \lambda_i e_i \quad \forall i \in [n]. \quad \lambda_1 = k.$$

$$A^2 e_1 + Ae_1 - (k-1)e_1 = Je_1,$$

$$\lambda_1^2 e_1 + \lambda_1 e_1 - (k-1)e_1 = ne_1,$$

$$\lambda_1^2 + \lambda_1 - (k-1) = n$$

$$k^2 + k - (k-1) = n = k^2 + 1 \quad \checkmark$$

$$\lambda_1 = k \text{ mult. 1}$$

$$\lambda_2 = \frac{-1+s}{2} \text{ mult. } m_2$$

$$\lambda_3 = \frac{-1-s}{2} \text{ mult. } m_3$$

$$i \in \mathbb{Z} \quad Je_i = 0 \quad \forall i \quad e_i \perp e_1.$$

$$e_i \perp \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$1 + m_2 + m_3 = n = k^2 + 1$$

$$m_2 + m_3 = k^2$$

$$\lambda_i^2 + \lambda_i - (k-1) = 0$$

$$\lambda_i = \frac{-1 \pm \sqrt{4k-3}}{2}$$

$$0 = \text{Tr}(A) = \frac{\lambda_1}{k} + m_2 \lambda_2 + m_3 \lambda_3$$

$$\lambda_1 = k \quad \lambda_{2,3} = \frac{-1 \pm s}{2} \quad \text{where}$$

$$s^2 = 4k-3$$

$$k = \frac{s^2+3}{4}$$

$$m_2 \lambda_2 + m_3 \lambda_3 = -k$$

$$m_2 \left( \frac{-1+s}{2} \right) + m_3 \left( \frac{-1-s}{2} \right) = -k$$

$$s(m_2 - m_3) = k^2 - 2k$$

$$m_2(-1+s) + m_3(-1-s) = -2k$$

$$= k(k-2)$$

$$m_2(1-s) + m_3(1+s) = 2k$$

$$s = \sqrt{4k-3}$$

$$\underbrace{(m_2 + m_3)}_{k^2} + s(m_2 - m_3) = 2k$$

If  $4k-3$  not a square, then  
 $s$  irrational

In all other cases,  $s \in \mathbb{Z}$ , so  
 $4k-3 = \text{square}$ .

then  $m_2 = m_3 \Rightarrow k(k-2) = 0$  (it doesn't really work...)  
 $\Rightarrow k = 2$

$$s \in \mathbb{Z}^+ (s \geq 0) \quad s(m_2 - m_3) = k^2 - 2k \quad k = \frac{s^2 + 3}{4}$$

$$k^2 - 2k - s(m_2 - m_3) = 0$$

$$\left(\frac{s^2 + 3}{4}\right)^2 - 2\left(\frac{s^2 + 3}{4}\right) - s(m_2 - m_3) = 0$$

$\times 16$

$$(s^2 + 3)^2 - 8(s^2 + 3) - 16s(m_2 - m_3) = 0$$

$$s^4 + 6s^2 + 9 - 8s^2 - 24 - 16s(m_2 - m_3) = 0$$

$$s^4 - 2s^2 - 16s(m_2 - m_3) - 15 = 0$$

$s$	$\frac{s^2 + 3}{4}$	note $-2, -16(m_2 - m_3), 0 \in \mathbb{Z}.$
$\pm 1$	1	$a_n = 1 \Rightarrow s \in \{\pm 1, \pm 3, \pm 5, \pm 15\}.$
$\pm 3$	3	(DO from beginning.)
$\pm 5$	7	
$\pm 15$	57	$\therefore k \in \{1, 2, 3, 7, 57\}.$ D

### Grad Student Panel

When did you start looking into grad school?

- K: 3rd yr. (always thinking about going).
- D: summer b/w 3rd / 4th yrs.
- Babai: consider what you want to go in grad yr.  
→ need to solve challenge prob in Babai's classes
- letters of recommendation from professors who know things about you to praise.

How do you make yourself known?

- B: Do well in classes ... did REUs outside of UChicago.  
(can be advantageous)
- K: Did REU for 3 yrs. - Peter May wrote letter, and prof. from Illinois Alg. spring quarter worked with over summer

- succeed in graduate course is a good sign.
- Babai: begin building a strategy of how you will be known by professors.
- D: most apps. need 3 recommendations - ideally - 3 strong letters from profs. who know you outside of class.
- Babai: 1 very strong letter and 2 OK letters is better - want to see you can do research + others formally.

3rd yr?

- K: taking a lot more classes...
- General GRE - like SAT (harder vocab)
- Math GRE - know all basic classes very well.
  - computation questions - fast! (integrals)
- Fall: think about what to write in your personal statement.
- application largely same between schools - fees per app
- selecting schools - look at faculty - ( $> \$100$ )
  - which have good faculty.
- P: prof helped pick McGill - named top # Heavily there.  
(grad ranking)
- Babai: find someone who is currently very active researcher and work with them.
- K: don't have to make these decisions before applying... post-admission, recruiting happens
- app due Dec - decisions Jan/Feb - final choice Apr.
- Babai: don't go w/o a plan B - consider schools that are really good and not necessarily on top list of graduate schools.
  - State schools, esp. in midwest, tend to be good.

- Don't bet on getting into a top school.
- K: State schools are good b/c have a lot more profs - good if you're undecided on field of research.
- Babai: Develop your own insights and show it  $\rightarrow$  be quick about clarifying things you don't understand.

### UChicago Grad (Dylan)

- 1<sup>st</sup> yr: all take same classes - exposure to fields
- 2<sup>nd</sup> yr: advisor - did combinatorics before.
- Babai: it's pretty easy to research profs. and see their research - if you feel affinity w/one - reach out to talk about work.
- B: asked recommends where to apply - geometry / topology

What about non-math profs?

- D: worked on materials research project - asked that prof for rec. instead of another math prof.
- Babai: trade off.

What are grad schools looking for?

Creativity - will have to create new mathematics eventually.  
 (Persistence, tolerance for failure, hardworking... good, but not enough)

Proofs don't come out linearly... more like a tree, where the proof is hiding in a leaf... you will trace out a path to the answer and will trace out the entire tree in the process.

"Doing math is 99% frustration."

## How to work on creativity?

- Babai - try to solve difficult probs in new ways.  
(solve a lot of problems) Push yourself.
- see problems in our worlds - 1 week problems  
5 min probs not indicative of your potential.

## DRP (Directed Reading Program)

- B: Morse theory (2nd yr) - might have been a bit advanced - 3rd / 4th yr good
  - K: sign up for class + get good student mentor → no grade / meet once a week.
  - give talk 1st week of next quarter.  
(similar to project part of PEU.)
  - letter of rec?
    - Babai: usually math problem ends up being good student doesn't know how to write about student - if can find words, will work.
- Or get good student to tell prof about you / write letter + prof puts name on it.

## Weapons of Math Destruction - Cathy O'Neil

- Algorithms perpetuating inequality.