Problem set 2

1. Prove that in the multiplication table of a group, every element appears exactly ones in each row and each column.

2. The points $A_1, \ldots, A_n$ form a regular polygon, inscribed in a circle with the center $O$. A point $X$ lies on the same circle. Prove that the images of the point $X$ under the symmetries with axes $OA_1, OA_2, \ldots, OA_n$ form a regular polygon.

3. Consider a regular polygon with vertices $A_1, A_2, \ldots, A_n$ and center $O$. Prove that
   \[ \overrightarrow{OA_1} + \overrightarrow{OA_2} + \ldots + \overrightarrow{OA_n} = 0. \]

4. Lagrange theorem. Let $G$ be a group of finite order. Prove that for every $g \in G$ we have
   \[ g^{|G|} = 1. \]

5. The Inclusion-Exclusion Principle
   Consider $N$ objects and some list $P_1, P_2, \ldots, P_n$ of their properties. Let $N_i$ be the number of objects satisfying $P_i$, $N_{ij}$, the number of objects satisfying $P_i$ and $P_j$, and so on. Prove that the number of objects satisfying none of these properties is equal to
   \[ N - \sum N_i + \sum_{i_1 < i_2} N_{i_1i_2} - \sum_{i_1 < i_2 < i_3} N_{i_1i_2i_3} + \ldots + (-1)^n N_{123\ldots n}. \]

6. Prove that if we remove two opposite corners from the chessboard, the board cannot be covered by dominoes (Each domino covers two neighboring cells of the chessboard.)

7. Chess Town
   (a) Consider an $m \times n$ rectangular grid: the Chess Town. It consists of $mn$ districts, divided by $n - 1$ horizontal and $m - 1$ vertical streets. What is the number of distinct shortest path on the grid, leading from the bottom left corner to the top right one?
   (b) What is the number of ways to draw a shortest path from the bottom left corner to the top right one lying below the diagonal connecting these corners? The path intersects the diagonal in two corner squares only.

8. Prove that a bounded figure in $\mathbb{R}^2$ cannot have more than one center of symmetry.

9. Prove in different ways that
   \[ \sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}. \]