Problem set 6

1. Prove that for every two nonintersecting circles $S_1$ and $S_2$ there exists an inversion, making them concentric.

2. Consider four circles such that $S_1$ and $S_3$ intersect both $S_2$ and $S_4$. Prove that if the points of intersection of $S_1$ with $S_2$ and $S_3$ with $S_4$ lie on the same circle or line, then the points of intersection of $S_1$ with $S_4$ and $S_2$ with $S_3$ lie on the same circle or line.

3. Steiner’s Chain. Suppose that there exists a chain of circles $S_1, S_2, \ldots, S_n$, such that $S_i$ is tangent to $S_{i+1}$ (and $S_n$ is tangent to $S_1$) and all $S_i$ are tangent to the two fixed circles $R_1$ and $R_2$. Prove that there exist infinitely many such chains.

4. (a) Prove that every Möbius transformation in $PGL_2(\mathbb{C})$ has one or two fixed points. The usual formulation of this fact is that there are two fixed points counted with multiplicity.

(b) Prove that a square of a Möbius transformation $t \mapsto \frac{at + b}{ct + d}$ is the identity if and only if $a + d = 0$.

5. (a) Suppose that for a Möbius transformation $f \in PGL_2(\mathbb{C})$ there exists a point $a$ such that $f(a) \neq a$, but $f(f(a)) = a$. Prove that $f$ is an involution.

(b) Prove that every Möbius transformation can be presented as a composition of at most three Möbius involutions.

6. Erdős-Szekeres theorem. Prove that for any $n, m \in \mathbb{N}$, every sequence of $nm + 1$ distinct real numbers contains an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $m + 1$. 
