# DISCRETE MATHEMATICS PROBLEMS. JUNE 20, 2002

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**Exercise 1.** Given $B \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, show that if $(\forall x)(xBx^t = 0)$ and $B = B^t$, then $B = 0$.

**Exercise 2.** If $A$ is an $n \times n$ orthogonal matrix (i. e., $AA^t = I$) and $\lambda$ is an eigenvalue of $A$, then $|\lambda| = 1$.

**Exercise 3.** Prove from first principles (i.e., without using the determinant) that the row rank of a matrix equals the column rank. *Hint:* Gaussian elimination.

**Definition 4.** The determinant rank of a matrix is the size of the largest non-singular submatrix. A square matrix is *non-singular* if its determinant is non-zero.

**Exercise 5.** Prove that the determinant rank of a matrix equals the row rank, which equals the column rank.

Recall Lindsey's lemma: If $A$ is an $a \times b$ submatrix of a Hadamard matrix, then $|\sum a_{ij}| \le \sqrt{abn}$.

**Exercise 6.** It was shown in class that the payoff for the Gale-Berlekamp switching game is $\le \sqrt{2}n^{3/2}$. Replace this with $\le (1 + o(1))n^{3/2}$.
*Hint:* Prime Number Theorem. Note: $a_n = o(1)$ is equivalent to $a_n \to 0$.

**Exercise 7.** We want to pick numbers from $1, \dots, n$ such that there is no 3-term arithmetic progression among the numbers selected. In class we gave an algorithm for doing this and getting $\Omega(\sqrt{n})$ numbers; now find an *explicit* subset of this size.

**Definition 8.** Given a $k$-colored matrix (i.e., one with entries $1, \dots, k$), a homogeneous submatrix is a submatrix all of whose entries are equal.

**Exercise 9.** Show that $(\forall k, \ell)(\exists n_0)$ such that if $n > n_0$, then every $k$-colored $n \times n$ matrix has a homogeneous $\ell \times \ell$ submatrix.
Prove this for $\ell = \Omega\left(\frac{\log n}{\log k}\right)$.
*Hint:* Focus on $k = 2$.

**Exercise 10.** As above, let $k = 2$, and show that $\ell = O(\log n)$; i.e., there exist $n\times n$ $(0, 1)$-matrices such that every homogeneous $\ell \times \ell$ submatrix satisfies $\ell = O(\log n)$.
*Hint:* Probabilistic method. (Flip coins to decide the entries.)

**Exercise 11.** Prove: if an $n \times n$ Hadamard matrix has an $\ell \times \ell$ homogeneous submatrix, then $\ell \le \sqrt{n}$.
*Hint:* Lindsey's Lemma

**OPEN QUESTION.** Construct an explicit family of $n \times n$ $(\pm 1)$ matrices such that all $\ell\times\ell$ homogeneous submatrices have $\ell \le n^{.49}$. A candidate is the Quadratic Residue matrix over $\mathbb{F}_p$, defined by $a_{i,j} = \left(\frac{i+j}{p}\right)$.

**Exercise 12.** If $q$ is a square ($q = p^{2k}$), then the Quadratic Residue matrix over $\mathbb{F}_q$ *does* have a homogeneous submatrix of size $\sqrt{q} \times \sqrt{q}$. Since Weil's Theorem does not distinguish between square and non-square prime powers, this suggests that it cannot be used to resolve the Open Question above.

**Exercise 13.** The Sylvester matrix $S_k$ (of size $2^k \times 2^k$) has $\sqrt{n} \times \sqrt{n}$ homogeneous submatrices.