Discrete Math, Fourth Problem Set (June 26)

REU 2003

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1 Linear Algebra

Remark 1.1. Recall that if we have a polynomial over $\mathbb{Z}$, $f(x) = a_n x^n + \cdots + a_0$, and a rational root p/q with $\gcd(p, q) = 1$, then $p \mid a_0$, and $q \mid a_n$.

Theorem 1.2. *If A is an $n \times k$ matrix, and B is a $k \times n$ matrix, then $Tr(AB) = Tr(BA)$.*

Corollary 1.3. *If A and B are $n \times n$ matrices, and $A \sim B$, then $Tr(A) = Tr(B)$.*

Proof: $\mathrm{Tr}(B) = \mathrm{Tr}((S^{-1}A)S) = \mathrm{Tr}(S(S^{-1}A)) = \mathrm{Tr}(A)$. □

We may also prove this by considering the characteristic polynomial of A. Let $f_A(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$. Then $c_{n-1} = -\mathrm{Tr}(A)$, $c_0 = (-1)^n \det(A)$, and generally:

$$c_k = (-1)^k \sum_{M \in \binom{n}{k}} \det(M),$$

where the sum is over all $k \times k$ symmetric minors of A. A symmetric minor is a submatrix symmetrically positioned with respect to the main diagonal, i. e., it has the same row numbers and column numbers. Since the characteristic polynomial is preserved under similarity, all such expressions are preserved, so specifically, the traces of similar matrices are equal.

Exercise 1.4. Prove that the trace is the sum of the eigenvalues (over $\mathbb{C}$). *Hint.* Prove that for any monic polynomial of degree n, the sum of the roots is the coefficient of x^{n-1}, times (-1).

Remark 1.5. Recall that every matrix over $\mathbb{C}$ is similar to an upper triangular matrix, and the diagonal entries of an upper triangular matrix are its eigenvalues. This gives an alternative proof of the fact that the trace of a square matrix is the sum of its eigenvalues.

2 Graphs

Definition 2.1. A **graph** is a (finite) set V of vertices, and a set E of edges, where an edge is an unordered pair of vertices.

Definition 2.2. We say that a pair of vertices v and w are **adjacent** $(x \sim y)$ if $\{v, w\} \in E$ and **non-adjacent** otherwise.

Definition 2.3. The **neighbors** of a vertex v are the vertices adjacent to v.

Definition 2.4. If $G = (V, E)$, and $H = (W, F)$ are graphs, then we say that a function $f : V \to H$ is an **isomorphism** if f preserves adjacency. That is, $x \sim y \iff f(x) \sim f(y)$. If an isomorphism exists between two graphs, then we say they are **isomorphic**.

Definition 2.5. The **degree** of a vertex is the number of its neighbors.

Definition 2.6. A **bipartite graph** is a graph (V, E) such that $V = V_1 \dot{\cup} V_2$, where each edge contains exactly one element from each V_i.

Definition 2.7. A **path** of length n in a graph is a sequence of distinct vertices, $(v_0, v_1, v_2, \ldots, v_n)$ where $\{v_{i-1}, v_i\}$ is an edge for all i.

Definition 2.8. A **walk** of length n in a graph is a sequence of (not necessarily distinct) vertices, $(v_0, v_1, v_2, \cdots v_n)$ where $\{v_i, v_{i+1}\}$ is an edge for all i.

Definition 2.9. A **cycle** of length $n \geq 3$ in a graph is a walk $(v_0, v_1, v_2, \ldots, v_n)$ where $v_0 = v_n$ but otherwise there are no repeated vertices.

Remark 2.10. Notice that a path of length n will have $n + 1$ vertices, and a cycle of length n will have n vertices.

Notation 2.11. P_n denotes the path on n vertices and C_n denotes the n-cycle (the cycle on n vertices.

Definition 2.12. K_n will be the **complete graph on n vertices**, which is the graph such that for all $v \neq w$, $\{v, w\}$ is an edge of K_n.

Definition 2.13. $K_{k,\ell}$ will be the **complete bipartite graph on $k + \ell$ vertices**, which is the graph $(V \dot{\cup} W, E)$ such that V and W have k and ℓ elements respectively, and $E = \{\{v, w\} : v \in V, w \in W\}$.

Definition 2.14. The **distance** between any two vertices is the length of the shortest path between them. The distance is infinite if no such path exists.

Definition 2.15. The **diameter** of a graph is the longest distance in the graph.

Definition 2.16. A graph is **connected** if there is a path between any two vertices.

Definition 2.17. The **complement** of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $E \cap \overline{E} = \emptyset$, and $E \dot{\cup} \overline{E}$ is the set of all $\binom{n}{2}$ pairs of vertices of G.

Remark 2.18. Notice that we have the following: $\overline{C_4} = 2K_2$, $\overline{C_5} = C_5$,$\overline{P_3} = P_3$.

Exercise 2.19. If $G \cong \overline{G}$ then $n \equiv 0, 1 \mod 4$.

Definition 2.20. The **girth** of a graph is the length of the shortest cycle.

Definition 2.21. A **tree** is a connected graph with no cycles.

Remark 2.22. A tree has infinite girth.

Exercise 2.23. A tree with n vertices has $n - 1$ edges.

Exercise 2.24. Are the two graphs from the Petersen's graph handout isomorphic?

Theorem 2.25 (Handshake theorem). $\sum_{v \in V} \deg(v) = 2|E|$.

Theorem 2.26. *If G is a regular graph of degree r, and girth$(G) \geq 5$, then $n \geq r^2 + 1$.*

In what cases is this bound tight, i.e. $n = r^2 + 1$? If $r = 1$, then K_2 satisfies $n = r^2 + 1$. For $r = 2$, C_5 satisfies this equation. For $r = 3$, we have Petersen's graph. For $r = 7$ there is a graph of degree r, girth 5, and $n = 50$ vertices called the "Hoffman-Singleton graph."

Theorem 2.27 (Hoffman-Singleton). *If a regular graph of degree r with $n = r^2 + 1$ vertices and girth at least 5 exists then $r \in \{1, 2, 3, 7, 57\}$.*

Remark 2.28. We have named such graphs with $r \in \{1, 2, 3, 7\}$. It is not known whether such a graph with $r = 57$ exists. It is, however, known that if such a graph exists, it cannot be quite as nice as the smaller ones: its automorphism group will not act transitively on its vertices, i.e., not all vertices will be equivalent under automorphisms (self-isomorphisms) of the graph (Aschbacher).

Exercise 2.29. Let G be a regular graph of degree r and of diameter ≤ 2.

(a) Prove: $n \leq r^2 + 1$.

(b) Prove: if $n = r^2 + 1$ then $r \in \{1, 2, 3, 7, 57\}$.

Definition 2.30. The **adjacency matrix**, $A_G = [a_{ij}]$, of a graph, G, is the $n \times n$ matrix such that $a_{ij} = 1$ if $\{i, j\}$ is an edge of G, and 0 otherwise, where n is the number of vertices.

Remark 2.31. Note that the diagonal and therefore the trace of the adjacency matrix is zero.

Theorem 2.32. *If a graph is r-regular, then r is an eigenvalue of its adjacency matrix.*

Proof: The all-ones vector is an eigenvector for the adjacency matrix with eigenvalue r. $\square$

Remark 2.33. A_G is symmetric. Recall that this and the spectral theorem imply that it is diagonalizable over $\mathbb{R}$ with an orthonormal eigenbasis.

Exercise 2.34. If G is a connected regular graph, then the multiplicity of r as an eigenvalue is one (it is a **simple** eigenvalue).

Exercise 2.35. If G is a regular graph of degree r then the multiplicity of r as an eigenvalue is the number of connected componenets of G.

Exercise 2.36. If G is a regular graph and λ is an eigenvalue of G (i. e., an eigenvalue of A_G) then $|\lambda| \leq r$.

Exercise 2.37. If G is a connected regular graph of degree r then $-r$ is an eigenvalue if and only if G is bipartite.

3 Discussion of Problem Sets

Definition 3.1. A subset A of $\mathbb{Z}$ is a **module** if it is closed under addition, subtraction, and contains zero.

Remark 3.2. Closure under addition technically follows from closure under subtraction, so is a redundant assumption in the above definition.

The following theorem follows from the fact that a subgroup of a cyclic group is necessarily cyclic. It is also the statement that $\mathbb{Z}$ is a PID.

Theorem 3.3. *All modules are of the form $d\mathbb{Z}$.*

Exercise 3.4. Consider two arithmetic progressions:

$$a_n = a_0 + nd_a,$$

$$b_n = b_0 + nd_b,$$

where $a_0, b_0, d_a, d_b \geq 0$. Prove: the two arithmetic progressions intersect if and only if $\gcd(d_a, d_b) \mid a_0 - b_0$.