Review of *Algorithmic Randomness and Complexity* by Downey and Hirschfeldt

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1 Introduction

The study of algorithmic randomness begins with an innocuous question, namely what is a random sequence? To illustrate this problem, the authors of *Algorithmic Randomness and Complexity* present the reader with the following two infinite sequences, one of which was generated via coin flips:

$$1011010111010101111000010101000101 11\ldots$$
$$1010101010101010101010101010101010\ldots$$

Intuitively it is obvious that that only the first of these two sequences could possibly be considered "random," yet probability theory does nothing to confirm this feeling: any series of coin flips generated by fair coins is equally likely to occur. Most likely, the reader arrived at an opinion based on one or more of the following heuristics:

A random sequence is unpredictable. A gambler cannot become wealthy by gambling on the bits of a random sequence.

A random sequence is incompressible. A sequence with compressible prefixes has some recognizable pattern, and patterns are not random.

A random sequence is typical. Random have no distinguishable properties except those which also belong to most other sequences.

All three of these notions turn out to characterize the same classes of sequences, and the first half of *Algorithmic Randomness and Complexity* explores this phenomenon in detail. One can rigorously formalize the three randomness paradigms in terms of algorithms, whence the field "algorthmic randomness" derives its name.

Over the past decade, algorithmic randomness has become a remarkably productive research area, both in terms of popularity and number of results. The field not only tries to reconcile various formalizations of randomness but also tries to understand the interaction between randomness content and computational power. In terms of Kolmogorov complexity and computability theory, one might say that highly random sequences contain a lot of information but not much useful information. On the other hand, this perspective oversimplifies the situation. For example, the halting probability, a real number equal to the probability that a randomly selected (prefix-free) computer program halts, admits a computable approximation from below and is Martin-Löf random yet contains enough information to decide whether or not an arbitrary computer program halts.

2 Summary

Downey and Hirschfeldt's book provides a comprehensive and detailed overview of the field of algorithmic randomness. The book collates the majority of results available at the time of publication, standardizes their notation, fills in literature gaps with folklore results and previously unpublished theorems, adds new proofs and results, and includes valuable historical perspective. This book serves as a go-to reference for the field, providing clear statements for theorems previously only accessible via ambiguous, missing, or contentious citations.

Although Downey and Hirschfeldt's book covers advanced topics, it is completely self-contained. The book starts out with a clear yet fast-paced 100-page introduction to computability theory, along the lines of Soare [5] and Odifreddi [3, 4], followed by a streamlined discussion on complexity of finite strings, including plain Kolmogorov complexity, prefix-free complexity, halting probabilities, and other complexity measures for finite strings (see Li and Vitányi's book [1] for an introduction). After the first four chapters on these topics, Downey and Hirschfeldt move on to areas not covered in other books, with the exception of Nies's book [2] which admits substantial overlap on some topics. Chapter 5 gives a gentle introduction to left-r.e. reals, or reals which have computable approximations from below, with some emphasis on presentations.

Chapters 6 and 7 introduce the central randomness notions of Martin-Löf randomness, computable randomness, Schnorr randomness, and Kurtz randomness, each of which are characterized in terms of the three paradigms mentioned earlier. Some of these characterizations are more appealing than

others, yet their formalizations are uniform: each characterization consists of definitions involving martingales, initial segment complexity, or effective sets of measure zero (or one). Martin-Löf randomness, being the most-studied randomness notion, receives a chapter all to itself and has earned its popularity due to various nice properties, including Miller and Yu's Ample Excess Lemma and van Lambalgen's Theorem. Martin-Löf randomness, however, lacks a nice characterization in terms of the unpredictability paradigm, a shortcoming which leads to a central open problem of this field: is Martin-Löf randomness the same as Kolmogorov-Loveland randomness?

Chapter 8, the longest non-introductory chapter of this book, contains core results relating randomness to information content via Turing degrees. Results here include:

- the Kučera-Gács Theorem, which says that any set is computable relative to some Martin-Löf random,
- a previously unpublished result from Kautz's thesis which says there is a Martin-Löf random such that every noncomputable set which is computable from it is Turing equivalent to a Martin-Löf random (this also follows from Demuth's Theorem),
- a theorem of Nies, Stephan, and Terwijn which says that the Martin-Löf random reals, computable random reals, and Schnorr random reals can be separated within the high Turing degrees but coincide outside of them, and finally
- a result of de Leeuw, Moore, Shannon, and Shapiro that if the Lebesgue measure of sequences X such that A is c.e. in X is nonzero, then A is itself c.e.

Chapters 9 and 10 investigate other measures of relative randomness and includes the elegant Kučera-Slaman Theorem which says that a real is left-c.e. and Martin-Löf random iff it is the weight of some universal prefix-free machine's domain.

I was especially pleased to find a clear exposition on the deep theorem of André Nies which states that the following statements are equivalent:

- X is K-trivial (the nth prefix of X contains no more information than the number n, which is as little as possible)
- X is low for K (X does not make it easier to compress strings), and
- X is low for random (X does not help with derandomization).

Chapter 11 is dedicated to discussion of this important result and K-trivials in general. For example, every K-trivial set is superlow, and therefore any noncomputable c.e. K-trivial set has incomplete Turing degree. There exists such a set with a single line construction which provides not only an injury-free but priority-free solution to Post's Problem. Randomness frequently simplifies ideas from recursion theory; what once required an involved construction to build a weak truth-table complete but non-truth-table complete set can now be done instead by choosing any left-c.e. Martin-Löf random real.

Chapter 13 discusses algorithmic dimension, which deals with sequences which are only "partially" random, as measured in various ways. Effective Hausdorff dimension, unlike its classical analogue in analysis, has multiple characterizations in terms of randomness which indicate that the concept is intuitive and robust. These characterizations include a definition in terms of open covers (like the classical definition), a definition in terms of martingales, and a definition in terms of Kolmogorov complexity. Prior publications asserted that these notions were equivalent but a cohesive explanation seemed missing. I appreciate Downey and Hirschfeldt's effort to piece these results together as well as their simple example showing that a sequence with full dimension need not be Martin-Löf random.

Downey and Hirschfeldt examine other topics as well, such as Kummer's Gap Theorem on the Kolmogorov complexity of c.e sets, and his theorem that the set of Kolmogorov random strings are truth-table complete.

3 Opinion

Anyone with some knowledge of computability theory and/or Kolmogorov complexity will enjoy browsing through this book, and anyone doing research in these areas will find this reference essential. The book is both well-written and well-organized, and having the results from this book as a systematic treatment brings one's attention to some facts which one might otherwise overlook or even fail to find in the literature. The authors meticulously attribute mathematical ideas with citations, and the index for the book is outstanding.

The authors of this book frequently argue proofs from primitive notions, as opposed to creating long chains of proofs which refer recursively back to previous results. They often improve on the notation of previous papers and sometimes simplify arguments found elsewhere in the literature. The result is a highly readable book.

While I am thrilled with both the writing and the topic of this book, the book itself is physically unwieldy at just under 900 pages. The authors completed *Algorithmic Randomness and Complexity* in the wake of Nies's groundbreaking result on K-trivials, and consequently many people chose to study lowness properties at that time. In keeping up with the field, Downey and Hirschfeldt wrote more than three chapters on this topic (Chapters 11, 12, 14, and some of 15), and I find the book slightly overweighted in this area. I would have preferred to see a bit more on relationship between randomness and differentiability, randomness and ergodic theory, randomness and numberings, sets of minimal indices, and integer-valued martingales, however some of these connections and topics were not yet known at the time of publication.

I give this books two thumbs up for making a large amount of fascinating new material accessible to the mathematics and computer science communities.

References

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