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ON THE MOMENTS OF RANDOM DETERMINANTS

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Mathematics is the art of giving the same name to different things.

-Henri Poincaré, in Science and Méthode, 1908.

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ABSTRACT

This thesis investigates the moments of random determinants. We give the background and related work on this topic and introduce the main tools used in finding formulas for the moments of determinants of random matrices. We then survey known results on the second and fourth moment of the determinant of a random matrix. After surveying the known results, we apply these techniques in an intricate way to analyze the sixth moment. Specifically, we find recurrence relations for the sixth moment which we use to derive the generating function. We then use this generating function to find an exact formula and the asymptotic behavior. Finally, we also give an alternative derivation of Zhurbenko's results on the second moment of the determinant of a random symmetric matrix.

CHAPTER 1

INTRODUCTION

Random matrix theory, which studies the properties of matrices with random entries, has found widespread applications across numerous fields, including physics, mathematics, engineering, and computer science. The determinant, a fundamental invariant of a matrix, plays a crucial role in understanding the properties of random matrices. In this thesis, we delve into the moments of random determinants, specifically focusing on the second, fourth, and sixth moments.

1.1 Prior Work on the Moments of Random Determinants and the Contributions of this Thesis

The $k-$th moment of a random determinant (i.e., the expected value of the kth power of the determinant of a random matrix) is only partially understood. Early works by Turán [1955] and Fortet [1951] found that the second moment of the determinant of an $n \times n$ matrix with mean 0 and variance 1 entries is $n!$. For Gaussian random matrices, Forsythe and Tukey [1952], Nyquist et al. [1954], and Prékopa [1967] showed that the kth moment of the determinant is $\prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}$. However, for other random matrices, no general formula for the $k-$th moment of the determinant is known.

For $n \times n$ random matrices whose entries have mean 0 and variance 1, the fourth moment of the determinant was determined by Nyquist et al. [1954]. Recently, the author and collaborators determined the sixth moment of the determinant (Beck et al. [2023]) and we present this analysis in this thesis.

In the study of moments of random determinants, the main idea employed by previous research is the reduction of the problem of computing these moments to counting the number of permutation tables subject to certain constraints. Permutation tables serve as a crucial tool in this context; the second moment of the random determinant can be easily obtained using them, as we will demonstrate in Section 3.1. For the fourth moment, Nyquist et al. [1954] employed recurrence relations and generating functions to analyze the number of permutation tables, which we will present in Section 3.2.

In the case of the sixth moment, which constitutes the primary focus of this thesis, we also utilize permutation tables. However, as we will mention in the following sections, unlike the cases with $k = 2$ or $k = 4$, the sign of each permutation table is not always positive. To address this challenge, we derive some structural results for permutation tables that enable us to obtain a formula for computing the sixth moment, including a generalization for $m_3 \neq 0$. Finally, we employ tools for analyzing generating functions developed by Borinsky [2018] to find the asymptotic behavior for the sixth moment, thereby significantly advancing our understanding of this problem.

For the case of symmetric random matrices, the generating function of the second moment of the determinant and the exact formula were found by Zhurbenko [1968]. From those results, Zhurbenko [1968] found the asymptotic behavior. In this thesis, we give an alternative derivation for the generating function and the exact formula.

1.2 Related Work

1.2.1 Generalizations

There have been several variations and generalizations of these questions. The first line is to replace the symmetric random variables with arbitrary variables. Zhurbenko [1968] started the investigation in this direction where he analyzed the second moment of random matrices whose entries is i.i.d from any distribution instead of a symmetric distribution.

Another generalization is generalizing these results for $p \times n$ matrices (where we consider $E\left[\det(MM^T)^{\frac{k}{2}}\right]$ rather than $E\left[\det(M)^k\right]$). Dembo [1989] obtained the formula for the case that $k = 2$.

For both of these two directions, Beck [2022] found the new results where he analyzed the fourth moment of the determinant of an $n \times n$ random matrix with independent entries from an arbitrary distribution and obtained the formula for $E\left[\det(MM^T)^{\frac{k}{2}}\right]$ for $k = 4$.

1.2.2 The Distribution of the Determinant

Apart from the moments, the distribution of the determinant of a random matrix has also been of great interest. Girko [1980, 1998] demonstrated that, under certain assumptions, the logarithm of the determinant obeys a central limit theorem. Nguyen and Vu [2014] later provided a simpler proof for a stronger version of this theorem. Following this line of research, Tao and Vu [2006a] obtained a nearly tight bound on the magnitude of determinant, that $|\det M_n| \geq \sqrt{n} e^{-29\sqrt{n \log n}}$.

Another line of inquiry has focused on the probability of a random $n \times n$ matrix with ± 1 entries being singular. In this line of work, Komlós [1967, 1968] was the first to prove that this probability is $o(1)$. Kahn et al. [1995] proved that this probability is at most $.999^n$, which was the first exponential upper bound. A series of works of Tao and Vu [2006b, 2007], Bourgain et al. [2010] improved this upper bound culminating in the work of Tikhomirov [2020] who proved an upper bound of $(\frac{1}{2} + o(1))^n$, which is tight. For the symmetric case, Costello et al. [2006] showed that the probability that a random symmetric matrix with ± 1 entries is singular is $o(1)$. For a more comprehensive survey of the combinatorial properties of random matrices, we refer the interested readers to the survey by Vu [2020].

CHAPTER 2

PRELIMINARIES

In this section, we first give the background and definitions that will be used throughout this thesis.

Definition 2.0.1. Given a distribution Ω, we define $\mathcal{M}_{n\times n}(\Omega)$ to be the distribution of $n \times n$ matrices where each entry is drawn independently from Ω.

Definition 2.0.2. Given a distribution Ω, we define m_k to be the kth moment of Ω, i.e.,

$$m_k = E_{x\sim\Omega}[x^k].$$

Definition 2.0.3. We define $f_k(n) = E_{M\sim\mathcal{M}_{n\times n}(\Omega)}\left[\det(M)^k\right]$ to be the expected value of the k-th power of the determinant of a random $n \times n$ matrix. Similarly, we define $p_k(n)$ to be the expected value of the k-th power of the permanent of a random $n \times n$ matrix.

Remark 2.0.4. Both $f_k(n)$ and $p_k(n)$ depend on the moments of Ω, but we write $f_k(n)$ and $p_k(n)$ rather than $f_{k,\Omega}(n)$ and $p_{k,\Omega}(n)$ for brevity.

2.1 Results

2.1.1 Moments of Determinants of Asymmetric Random Matrices

The first work analyzing the determinant of random matrices dates back to Fortet [1951] and Turán [1955], where Turán observed that the second moment of the determinant of an $n \times n$ matrix (where the entries have mean 0 and variance 1) is $n!$, and he also obtained the explicit formula for computing the fourth moment of the determinant of a random matrix where each entry is $\{-1, +1\}$. Later, Nyquist et al. [1954] showed the following result on the fourth moment.

Theorem 2.1.1. *$f_4(n) = n! y_n$ where y_n obeys the recurrence relation*

$$y_n = (n + m_4 - 1)y_{n-1} + (3 - m_4)(n - 1)y_{n-2}.$$

where $y_0 = 1$ and $y_1 = m_4$.

They further observed that if we take the generating function $Y(t) = \sum_{t=0}^{\infty} \frac{y_n t^n}{n!}$ then $Y(t) = (1-t)^{-3} e^{(m_4-3)t}$. From this generating function, they found the equation

$$f_4(n) = n! y_n = \frac{(n!)^2}{2} \sum_{k=0}^{n} \frac{(n-k+1)(n-k+2)}{k!} (m_4 - 3)^k.$$

To prove their results, Nyquist et al. [1954] counted $4 \times n$ tables with certain properties. As we describe in Section 3.3, we use the same general approach for the sixth moment though our analysis is considerably more intricate. Using the techniques mentioned above and combined with generating functions, we obtained the following results for the sixth moment of random determinants.

Theorem 2.1.2. *For any distribution Ω such that $m_1 = m_3 = 0$ and $m_2 = 1$, the formal generating function $F_6(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_6(n)$ for $f_6(n)$ is*

$$F_6(t) = \frac{e^{t(m_6 - 15m_4 + 30)}}{48\,(1 + 3t - m_4 t)^{15}} \sum_{i=0}^{\infty} \frac{(1+i)(2+i)(4+i)! t^i}{(1 + 3t - m_4 t)^{\,3i}}.$$

Performing Taylor expansion of this generating function, we get the formula for computing the sixth moment of random determinants, namely:

Corollary 2.1.3. *For any distribution Ω such that $m_1 = m_3 = 0$ and $m_2 = 1$,*

$$f_6(n) = (n!)^2 \sum_{j=0}^{n} \sum_{i=0}^{j} \frac{(1{+}i)(2{+}i)(4{+}i)!}{48(n-j)!} \binom{14{+}j{+}2i}{j-i} (m_6 - 15m_4 + 30)^{n-j} (m_4 - 3)^{j-i}.$$

Remark 2.1.4. If $m_2 \neq 1$ then we can scale the distribution Ω by $\frac{1}{\sqrt{m_2}}$ (which changes the determinant of matrices in $\mathcal{M}_{n\times n}(\Omega)$ by a factor of $\left(\frac{1}{\sqrt{m_2}}\right)^n$) and then apply the result in Corollary 2.1.3.

Remark 2.1.5. If $\Omega = N(0,1)$ then $m_4 = 3$ and $m_6 = 15$ so $f_6(n) = P_n = \frac{n!(n+2)!(n+4)!}{48}$, which is a special case of the result that $f_k(n) = \prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}$ when $\Omega = N(0,1)$ and k is even.

Another generalization is when $m_3 \neq 0$.

Theorem 2.1.6. *For any distribution Ω such that $m_1 = 0$ and $m_2 = 1$, the formal generating function $F_6(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_6(n)$ for $f_6(n)$ is*

$$F_6(t) = \left(1 + m_3^2 t\right)^{10} \frac{e^{t(m_6 - 10m_3^2 - 15m_4 + 30)}}{48\left(1 + 3t - m_4 t\right)^{15}} \sum_{i=0}^{\infty} \frac{(1+i)(2+i)(4+i)!t^i}{\left(1 + 3t - m_4 t\right)^{3i}}.$$

Corollary 2.1.7. *For any distribution Ω such that $m_1 = 0$ and $m_2 = 1$,*

$$f_6(n) = (n!)^2 \sum_{j=0}^{n} \sum_{i=0}^{j} \sum_{k=0}^{n-j} \frac{(1+i)(2+i)(4+i)!}{48(n-j-k)!} \binom{10}{k} \binom{14+j+2i}{j-i} q_6^{n-j-k} q_4^{j-i} q_3^k,$$

where

$$q_6 = m_6 - 10m_3^2 - 15m_4 + 30, \qquad q_4 = m_4 - 3, \qquad q_3 = m_3^2.$$

Below, we show the values of $f_k(n)$ and $p_k(n)$ when $\Omega = \{-1, 1\}$ for small values of k and n. We note that when $\Omega = \{-1, 1\}$, $f_4(n)$ is the integer sequence A052127 in the On-Line Encyclopedia of Integer Sequences Costello [2007]. In the entry for this integer sequence, it is noted that $f_4(n) \sim (n!)^2 \frac{(n^2+7n+10)}{(2e^2)}$ as $n \to \infty$.

Remark 2.1.8. Note that for all $n \in \mathbb{N}$, $p_2(n) = f_2(n)$ and $p_4(n) = f_4(n)$. However, for $n \geq 3$, $p_6(n) > f_6(n)$.

n	$f_2(n)$	$f_4(n)$	$f_6(n)$	$p_2(n)$	$p_4(n)$	$p_6(n)$
1	1	1	1	1	1	1
2	2	8	32	2	8	32
3	6	96	1536	6	96	2976
4	24	2112	282624	24	2112	513024
5	120	68160	66846720	120	68160	157854720

Table 2.1: Values of $f_k(n)$ and $p_k(n)$ for $\Omega = \{-1, 1\}$, $k \leq 6$, and $n \leq 5$

We can describe the asymptotic behavior of f_6 using the following asymptotic expansion.

Theorem 2.1.9. *For all $R \in \mathbb{N} \cup \{0\}$,*

$$f_6(n) = \frac{e^{3q_4}(n!)^2}{48} \left(\sum_{k=0}^{R} c_k (n+6-k)! \right) \pm O\left((n!)^2 (n+6-R-1)! \right),$$

where the coefficients c_k are the Taylor expansion coefficients of the function $C(t) = \sum_{k \geq 0} c_k t^k$,

$$C(t) = e^{(q_6 - 3q_4^2)t + q_4^3 t^2} (1 + q_3 t)^{10} \Big(1 - 2(3q_4 + 4)t + 3\left(5q_4^2 + 8q_4 + 4\right)t^2 - 4\left(q_4^2(5q_4 + 6)\right)t^3 + q_4^3(15q_4 + 8)t^4 - 6q_4^5 t^5 + q_4^6 t^6 \Big).$$

Remark 2.1.10. For the first terms in the expansion, we have

$$f_6(n) \sim \frac{e^{3m_4 - 9}}{48}(n!)^3 \Big(n^6 + \left(m_6 - 3m_4^2 - 3m_4 + 34\right) n^5 + \frac{1}{2}\Big(m_6^2 - 10m_3^4 + 9m_4^4 + 20m_4^3 - 183m_4^2 - 126m_4 - 6m_4^2 m_6 - 6m_4 m_6 + 56m_6 + 905\Big) n^4 + \cdots \Big)$$

Remark 2.1.11. Note that when $\Omega = \{-1, 1\}$, as $n \to \infty$,

$$f_6(n) \sim \frac{(n!)^3}{48e^6} \left(n^6 + 29n^5 + 335n^4 + \frac{5861n^3}{3} + \frac{17944n^2}{3} + \frac{44036n}{5} + \frac{167536}{45} - \frac{210176}{63n} \right).$$

2.1.2 Moments of Determinants of Symmetric Random Matrices

Zhurbenko [1968] obtained both the exact formula and asymptotic behavior of the second moments of determinants of symmetric random matrices. In this thesis, we give an alternative derivation for the exact formula based on the counting of certain kind of graphs, which is similar to the permutation tables used for the asymmetric case.

Definition 2.1.12. We define $f_k^{sym}(n) = E_{M \sim \mathcal{M}_{n \times n}(\Omega)}\left[\det(M)^k\right]$ to be the expected value of the k-th power of the determinant of a random $n \times n$ symmetric matrix, where M_{ij} are drawn independently from each other only when $i \le j$, for the rest the values are copied across the main diagonal, i.e. $M_{ij} = M_{ji}$. Similarly, we define $p_k^{sym}(n)$ to be the expected value of the k-th power of the permanent of a random $n \times n$ symmetric matrix.

Theorem 2.1.13. *For any distribution Ω such that $m_1 = 0$ and $m_2 = 1$,*

$$f_2^{sym}(n) = n! \sum_{n=0}^{\infty} \sum_{0 \le q \le \lfloor \frac{n}{2} \rfloor} \prod_{i=1}^{2q} \frac{2i-1}{2i} \sum_{0 \le p \le n-2q} (n-2q-p+1) \sum_{k=\lceil \frac{p}{2} \rceil}^{p} \frac{(-1)^k}{(p-k)!(2k-p)!} \left(\frac{3-m_4}{2}\right)^{p-k}.$$

Zhurbenko [1968] showed the following asymptotic behavior of $f_2^{sym}(n)$.

Theorem 2.1.14. *For any distribution Ω such that $m_1 = 0$ and $m_2 = 1$,*

$$f_2^{sym}(n) = C_n n^{\frac{3}{2}} n!$$

where $\lim_{n \to \infty} C_n = \frac{4\sqrt{2\pi}e^{-2}}{3}$.

2.1.3 Gaussian Entries

When each entry of the random matrix is sampled from the standard normal distribution, there is a general formula for the kth moment.

n	$f_2^{sym}(n)$	$p_2^{sym}(n)$
1	1	1
2	2	2
3	8	8
4	44	44
5	244	244
6	1744	1744
7	13768	13768
8	127952	127952

Table 2.2: Values of $f_2^{sym}(n)$ for $\Omega = \{-1, 1\}$ and $n \leq 8$

Definition 2.1.15. We define $f_k^{SN}(n) = E_{M \sim \mathcal{M}_{n\times n}(\Omega)}\left[\det(M)^k\right]$ to be the expected value of the k-th power of the determinant of a random $n \times n$ matrix, where $M_{ij} \sim \mathcal{N}(0,1)$.

Theorem 2.1.16. *When k is even,*

$$f_k^{SN}(n) = \prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}.$$

Remark 2.1.17. When k is odd, $f_k^{SN}(n) = 0$.

2.2 Permutation Tables and Graphs

To prove the results mentioned above, we need a few definitions and a key lemma.

Definition 2.2.1. Given natural numbers k and n where k is even, we define an even $k \times n$ table to be a $k \times n$ table where each row is a permutation of $[n]$ and each column contains each number an even number of times. We define $T_{k,n}$ to be the set of all even $k \times n$ tables.

Definition 2.2.2. Given an even table t of size $k \times n$, we define its sign $\text{sgn}(t)$ to be the product of the signs of its rows, which are permutations of $[n]$.

Definition 2.2.3. Given a column c where each element is in $[n]$, we define its weight $w(c)$

to be

$$w(c) = \prod_{j=1}^{n} m_{\text{\# of times } j \text{ appears in column c}}.$$

For even $6 \times n$ tables, we say that a column is a 6-column if it contains some number 6 times, a 4-column if it contains one number four times and another number two times, and a 2-column if it contains three different numbers two times. Observe that the weight of a 6-column is m_6, the weight of a 4-column is m_4, and the weight of a 2-column is m_2.

Definition 2.2.4. Given an even $k \times n$ table t, we define its weight $w(t)$ to be the product of the weights of its columns.

We can use the following proposition to reduce the problem of finding the sixth moment of a random determinant to a combinatorial problem.

Proposition 2.2.5. *For all even $k \in \mathbb{N}$, $f_k(n) = \sum_{t \in T_{k,n}} \text{sgn}(t)w(t)$ and $p_k(n) = \sum_{t \in T_{k,n}} w(t)$.*

Proof. We observe that

$$f_k(n) = E_{A \sim \mathcal{M}_{n \times n}(\Omega)} \left[\sum_{\pi_1, \pi_2, \ldots, \pi_k \in S_n} \left(\prod_{i=1}^{k} \text{sgn}(\pi_i) \right) \prod_{p=1}^{n} \left(\prod_{q=1}^{k} A_{p,\pi_q(p)} \right) \right]$$

and

$$p_k(n) = E_{A \sim \mathcal{M}_{n \times n}(\Omega)} \left[\sum_{\pi_1, \pi_2, \ldots, \pi_k \in S_n} \prod_{p=1}^{n} \left(\prod_{q=1}^{k} A_{p,\pi_q(p)} \right) \right].$$

For each $p \in [n]$, we have that $E_{A \sim \mathcal{M}_{n \times n}(\Omega)}[\prod_{q=1}^{k} A_{p,\pi_q(p)}] = w(p)$ (i.e., the weight of column p), so $f_k(n) = \sum_{t \in T_{k,n}} \text{sgn}(t)w(t)$ and $p_k(n) = \sum_{t \in T_{k,n}} w(t)$, as needed. □

Thus, computing the kth moment of a random determinant is equivalent to summing the signed weights of all even tables of size $k \times n$.

Corollary 2.2.6. *If Ω is the uniform Bernoulli distribution (i.e., the uniform distribution on $\{-1, 1\}$) then $f_k(n) = \sum_{t \in T_{k,n}} \text{sgn}(t)$ and $p_k(n) = |T_{k,n}|$.*

Corollary 2.2.7. *If $k = 2$, $k = 4$, or $n \leq 2$ then $p_k(n) = f_k(n)$. If $n \geq 3$, $k \geq 6$, and k is even then $p_k(n) > f_k(n)$.*

To analyze $f_6(n)$, it is useful to consider tables together with pairings of identical elements in each column.

Definition 2.2.8. Given an even $k \times n$ table t, we define a pairing P on t to be a set of matchings $\{M_i : i \in [n]\}$, one for each column, where each matching M_i pairs up identical elements of column i. We define $\mathcal{P}(t)$ to be the set of all pairings on t.

Example 2.2.9. The table on the left below is an even 6×4 table with 27 possible *pairings*. The table on the right shows one of the 27 possible parings.

1	2	4	3
1	2	4	3
1	3	4	2
1	3	4	2
2	4	1	3
2	4	1	3

1	2	4	3
1	2	4	3
1	3	4	2
1	3	4	2
2	4	1	3
2	4	1	3

Table 2.3: Tables shows pairing

Proposition 2.2.10. *For each even $6 \times n$ table t,*

$$|\mathcal{P}(t)| = 15^{\text{\# of 6−columns in t}} 3^{\text{\# of 4−columns in t}}.$$

Definition 2.2.11. We define $P_n = \sum_{t \in T_{k,n}} \operatorname{sgn}(t) |\mathcal{P}(t)|$.

It turns out that P_n can be easily computed and this is crucial for our results.

Lemma 2.2.12. *For all $n \in \mathbb{N}$, $P_n = n(n+2)(n+4)P_{n-1}$ where $P_0 = 1$.*

This lemma follows from the fact that when $\Omega = N(0,1)$ and k is even, the kth moment of the determinant is $\prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}$. We give a direct proof of this lemma in Section 3.3.3.

Note that $P_n = \sum_{t \in T_{k,n}} \text{sgn}(t) 15^{\text{\# of 6-columns in t}} 3^{\text{\# of 4-columns in t}}$ while $f_6(n) = \sum_{t \in T_{k,n}} \text{sgn}(t) m_6^{\text{\# of 6-columns in t}} m_4^{\text{\# of 4-columns in t}}$. If $\Omega = N(0,1)$ (or we at least have that $m_2 = 1$, $m_4 = 3$, and $m_6 = 15$) then $f_6(n) = P_n$. In the next section, we show how to handle other distributions Ω using inclusion/exclusion.

CHAPTER 3

MOMENTS OF DETERMINANTS OF RANDOM MATRICES

3.1 Second Moment

First we present the following result due to Turán [1955] as a warm-up for an application of the techniques of counting tables.

Theorem 3.1.1. *For any distribution Ω such that $m_1 = 0$ and $m_2 = 1$,*

$$f_2(n) = n!.$$

Proof. By 2.2.5, $f_2(n) = \sum_{t \in T_2(n)} \operatorname{sgn}(t) w(t)$. For each table t in $T_2(n)$, we know that each column must have two identical numbers, so $\operatorname{sgn}(t)$ is always $+$ and each of them has $w(t) = 1$. Since there are $n!$ different choices for the first row and once the first row is selected the second row is determined automatically. We have $|T_2(n)| = n!$. □

3.2 Fourth Moment

The following result is based on the proof by Nyquist et al. [1954].

Theorem 3.2.1. *For any distribution Ω such that $m_1 = 0$ and $m_2 = 1$, $f_4(n) = n! y_n$ where y_n obeys the recurrence relation*

$$y_n = (n + m_4 - 1) y_{n-1} + (3 - m_4)(n-1) y_{n-2}.$$

where $y_0 = 1$ and $y_1 = m_4$.

Proof. By the same reasoning as 3.1.1, using 2.2.5, $f_4(n)$ is the number of $4 \times n$ tables which elements appear either two or four times in one column. For each of the tables, the contribution of an element to the overall weight is m_4 if this element appears in a 4-column and is unit if it appears in two 2-columns.

For the sign of each table, we show it is always positive by induction. The base case is clear that when $n = 1$, there is only possible table. Now assume that all even tables with size $4 \times (n-1)$ are always positive. Then for any even table t with size $4 \times n$, we denote the rows of t as r_1, r_2, r_3 and r_4, and denote the column position of elements n in t as c_1, c_2, c_3 and c_4. We know t is an even table, so without of loss generality, we assume that $c_1 = c_2$ and $c_3 = c_4$. Denote the elements in $n-$th column of t as a pair of a and a pair of b where it's possible that $a = b$. Then by symmetry, there are two possible arrangements of t :

$$\begin{Bmatrix} ... & n & ... & ... & ... & a \\ ... & n & ... & ... & ... & a \\ ... & ... & ... & n & ... & b \\ ... & ... & ... & n & ... & b \end{Bmatrix}, \begin{Bmatrix} ... & n & ... & d & ... & a \\ ... & n & ... & d & ... & b \\ ... & c & ... & n & ... & a \\ ... & c & ... & n & ... & b \end{Bmatrix}$$

Note here we assume that $a \neq b \neq n$. When $a = n$ or $b = n$, we just decrease the number of swaps by 2. For the first case, we just swap the position of all n so that they are all in the last column. Then the first $n - 1$ columns form a $4 \times (n - 1)$ table t'. Since in this case, we made 4 swaps, the sign of t is equal to t', which, by the inductive hypothesis, is positive. For the second case, we can make eight swaps to get a $4 \times (n - 1)$ table. If $a = b$, the number of swaps in this case is 4. By proving these two cases, we have shown that the sign of $4 \times n$ table is always positive.

To compute $f_4(n)$, we first fix the first row in natural order so that

$$f_4(n) = n! * y_n.$$

Here, y_n is a function of the total weight of $4 \times n$ tables which elements appear either two or four times in one column but with the first row fixed to be the natural order.

For tables in $T_4(n)$ where the first row is fixed to be the natural order, we split them into two groups based on whether the element n is in a 4-column or two 2-columns. For the tables in the group of the first case, we can see the contribution is simply $y_{n-1}m_4$. For the latter case, we know one of the 2-column with n is the last column because we have fixed the first row, and we have $n-1$ choices for the other 2-column. Besides, we also need to choose which rows in the last column are not n, which gives us $\binom{3}{2} = 3$ choices.

Without loss of generality, we may assume the last two columns to be

$$\begin{Bmatrix} n-1 & n \\ n-1 & n \\ n & x \\ n & x \end{Bmatrix}.$$

We denote the contribution of this case as $z(n)$. In this case, x is an element that cannot be n. So among all the $n-1$ choices for x, if x is selected to be $n-1$, then the contribution of the case is y_{n-2}. If x is any of the other $n-2$ elements, the contribution is z_{n-1}. Therefore,

$$y(n) = m_4 y_{n-1} + 3(n-1)z_n,$$

$$z(n) = y_{n-2} + (n-2)z_{n-1}.$$

By plugging in z_n to y_n,

$$y_n = (m_4 + n - 1)y_{n-1} + (n-1)(3 - m_4)y_{n-2}.$$

And it's easy to see the following base case: $y_0 = 1$ and $y_1 = m_4$. □

In order to solve the recurrence formula, we use the exponential generating function, where $Y(t) = \sum_{n\geq 0} \frac{y_n t^n}{n!}$. This recurrence gives us the following differential equation:

$$Y'(t) = tY'(t) + m_4 * Y(t) + (3 - m_4)tY(t).$$

By solving this differential equation, we get

$$Y(t) = (1-t)^{-3} e^{t(m_4-3)}.$$

Which leads us the following explicit form for $f_4(n)$,

$$f_4(n) = \frac{(n!)^2}{2} \sum_{k=0}^{n} \frac{(n-k+1)(n-k+2)}{k!} (m_4 - 3)^k.$$

3.3 Sixth Moment

Before preceding to the proof of Theorem 2.1.2, we first prove the following result on the sixth moment of random determinants.

Lemma 3.3.1. *For any distribution Ω such that $m_1 = m_3 = 0$ and $m_2 = 1$,*

$$f_6(n) = \sum_{j=0}^{n} \sum_{a=0}^{j} \binom{n}{j} \binom{j}{a} (m_6 - 15)^a (m_4 - 3)^{(j-a)} D_{n,a,j-a}.$$

$$D_{n,a,b} = \left(\prod_{j=0}^{a+b-1} (n-j) \right) \left(\sum_{i=0}^{b} \binom{b}{i} C_i H_{n,b-i,a,b} \right) P_{n-a-b}.$$

$P_n = n(n+2)(n+4)P_{n-1}$ *where* $P_0 = 1$*. Equivalently,* $P_n = \dfrac{n!(n+2)!(n+4)!}{2!4!}$.

$C_n = (n-1)(C_{n-1} + 15C_{n-2})$ *where* $C_0 = 1$ *and* $C_1 = 0$.

$$H_{n,j,a,b} = \sum_{x=1}^{j} \frac{\binom{j-1}{x-1}}{x!} j! \prod_{y=0}^{x-1} (3(n-a-b) - y).$$

Proof. The idea behind the proof is as follows. We consider the tables where we know that some set $A \subseteq [n]$ of elements appear six times in a 6-column and another set $B \subseteq [n] \setminus A$ of elements appear four times in a 4-column and two times in a different column. We do not know whether the elements in $[n] \setminus (A \cup B)$ appear six times in a 6-column, appear four times in a 4-column and two times in a different column, or appear two times in three different columns. We consider these tables together with pairings for the columns which are unaccounted for by A and B (i.e., the columns which don't contain six of the same element in A or four of the same element in B).

To obtain $f_6(n)$, we compute the contribution from each A and B and then take an appropriate linear combination of these contributions so that the contribution from each individual table t is $\text{sgn}(t)w(t)$.

Definition 3.3.2. Given $A \subseteq [n]$ and $B \subseteq [n] \setminus A$, we define $D_{n,A,B}$ to be the set of tables in $T_{6,n}$ such that the elements in A appear six times in a 6-column and the elements in B appear four times in a 4-column and two times in a different column.

For each $t \in D_{n,A,B}$, we define $\mathcal{P}_{A,B}(t)$ to be the set of pairings on t where we exclude the 6-columns which contain six of the same element in A and the 4-columns which include four of the same element in B.

By symmetry, the contribution from each $D_{n,A,B}$ only depends on $|A|$ and $|B|$.

Definition 3.3.3. Given $n, a, b \in \mathbb{N} \cup \{0\}$ such that $a + b \leq n$, we define $D_{n,a,b}$ to be

$$D_{n,a,b} = \sum_{t \in D_{n,A,B}} \text{sgn}(t)|\mathcal{P}_{A,B}(t)|$$

where $A \subseteq [n]$, $B \subseteq [n] \setminus A$, $|A| = a$, and $|B| = b$.

Lemma 3.3.4. *For all* $n \in \mathbb{N} \cup \{0\}$, $f_6(n) = \sum_{j=0}^{n} \sum_{a=0}^{j} \binom{n}{j} \binom{j}{a} (m_6 - 15)^a (m_4 - 3)^{(j-a)} D_{n,a,j-a}$.

Proof. Observe that

$$\sum_{j=0}^{n}\sum_{a=0}^{j}\binom{n}{j}\binom{j}{a}(m_6-15)^a(m_4-3)^{(j-a)}D_{n,a,j-a}$$
$$=\sum_{A\subseteq[n]}\sum_{B\subseteq[n]\setminus A}\sum_{t\in D_{n,A,B}}(m_6-15)^{|A|}(m_4-3)^{|B|}\operatorname{sgn}(t)|\mathcal{P}_{A,B}(t)|.$$

Given a table $t \in T_{6,n}$, let A' be the set of elements in $[n]$ which appear six times in a 6-column of t and let B' be the set of element which appear four times in a 4-column of t. Now consider the contribution from t in

$$\sum_{A\subseteq[n]}\sum_{B\subseteq[n]\setminus A}\sum_{t\in D_{n,A,B}}(m_6-15)^{|A|}(m_4-3)^{|B|}\operatorname{sgn}(t)|\mathcal{P}_{A,B}(t)|.$$

We have that whenever $A \subseteq A'$ and $B \subseteq B'$, $t \in D_{n,A,B}$ and $|\mathcal{P}_{A,B}(t)| = 15^{|A'\setminus A|}3^{|B'\setminus B|}$. Thus, the contribution from t is

$$\sum_{A\subseteq A'}\sum_{B\subseteq B'}(m_6-15)^{|A|}(m_4-3)^{|B|}15^{|A'\setminus A|}3^{|B'\setminus B|}\operatorname{sgn}(t)={m_6}^{|A'|}{m_4}^{|B'|}\operatorname{sgn}(t)=\operatorname{sgn}(t)w(t).$$

This implies that

$$\sum_{j=0}^{n}\sum_{a=0}^{j}\binom{n}{j}\binom{j}{a}(m_6-15)^a(m_4-3)^{(j-a)}D_{n,a,j-a}=\sum_{t\in T_{6,n}}\operatorname{sgn}(t)w(t)=f_6(n)$$

as needed. □

We now compute $D_{n,a,b}$.

Lemma 3.3.5. *For all $n, a, b \in \mathbb{N} \cup \{0\}$ such that $a + b \leq n$,*

$$D_{n,a,b}=\left(\prod_{j=0}^{a+b-1}(n-j)\right)\left(\sum_{i=0}^{b}\binom{b}{i}C_iH_{n,b-i,a,b}\right)P_{n-a-b}$$

1	2	3	4	5
1	2	3	4	5
1	2	5	3	4
1	2	5	3	4
2	1	3	4	5
2	1	3	4	5

h Figure 3.1: A 6×5 table $t \in D_{5,\emptyset,[5]}$

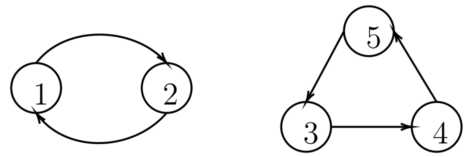

Figure 3.2: The associated $G(t)$

where C_n *is given by the recurrence relation* $C_n = (n-1)(C_{n-1} + 15C_{n-2})$, $C_0 = 1$, $C_1 = 0$ *and*

$$H_{n,j,a,b} = \sum_{x=1}^{j} \frac{\binom{j-1}{x-1}}{x!} j! \prod_{y=0}^{x-1} (3(n-a-b)-y).$$

Proof. To prove this lemma, we group the tables in $D_{n,A,B}$ based on the structure of the 4-columns containing four of the same element of B.

Definition 3.3.6. Given a table $t \in D_{n,A,B}$, we define the directed graph $G(t)$ to be the graph with vertices $B \cup \{center\}$ and the following edges. For each $b \in B$,

1. If there is a $b' \in B \setminus \{b\}$ such that there are two b in the 4-column containing four b' then we add an edge from b to b'.

2. If there is no such b' then we add an edge from b to center.

Proposition 3.3.7. *For all* $t \in D_{n,A,B}$, $G(t)$ *has the following properties.*

1. *For all* $b \in B$, $\deg^+(b) = 1$ *and* $\deg^-(b) \leq 1$.

2. $\deg^+(\text{center}) = 0$.

Corollary 3.3.8. *For all* $t \in D_{n,A,B}$, $G(t)$ *consists of directed cycles and paths which end at* center, *all of which are disjoint except for their common endpoint.*

Example 3.3.9. This example shows the correspondence between a table t and $G(t)$.

We now consider how many ways there are to start with a table $t \in T_{6,n-a-b}$ together with a pairing $P \in \mathcal{P}$ and construct a table $t' \in D_{n,A,B}$ (we will automatically have that the sign of t and the pairing P is preserved). Before giving the entire analysis, we describe the parts of the analysis corresponding to the cycles and paths of $G(t')$ as these are the trickiest parts of the analysis.

Definition 3.3.10. Define C_n to be the number of tables $t \in D_{n,\emptyset,[n]}$ such that $G(t)$ consists of directed cycles and for each $i \in [n]$, column i contains four i.

Lemma 3.3.11. *For all $n \geq 2$, $C_n = (n-1)(C_{n-1} + 15C_{n-2})$ where $C_0 = 1$ and $C_1 = 0$.*

Proof. Consider a vertex $i \in [n]$. $G(t)$ contains an edge from i to j for some $j \in [n] \setminus \{i\}$. Note that there are $n-1$ possibilities for j. We now have two cases. The first case is that there is an edge from j to a vertex $k \in [n] \setminus \{i, j\}$. In this case, we can remove the vertex j and the edges $(i, j), (j, k)$ and add an edge from i to k. The number of possibilities for this case (for a fixed j) is C_{n-1}. The second case is that there is an edge from j back to i, i.e., i and j are in a directed cycle of length 2. The number of possibilities for this case (for a fixed j) is $15C_{n-2}$ as there are 15 possibilities for which rows of column i contain j and there are C_{n-2} possibilities for the remaining $n-2$ columns.

Adding these two cases together and summing over all possible $j \in [n] \setminus \{i\}$, we have that $C_n = (n-1)(C_{n-1} + 15C_{n-2})$, as needed. □

To handle paths, we first count the number of possible graphs $G(t)$ with a given number of paths to center and no cycles with the following lemma.

Lemma 3.3.12. *Let $B' \subseteq [n]$ and take $j = |B'|$. For all $x \in [j]$, there are $\frac{\binom{j-1}{x-1}}{x!} j!$ possible graphs $G(t)$ on the vertices $B' \cup \{\text{center}\}$ which consist of x disjoint paths to* center *and no cycles.*

Proof. We can specify each such graph as follows:

1. Choose an ordering for the elements of B'. There are $j!$ possibilities for this ordering.

2. Choose the x paths by putting $x-1$ dividing lines among the elements of B'. Since each path must have at least one vertex in B', there are $\binom{j-1}{x-1}$ possibilities for this.

However, if we do this, each graph is counted $x!$ times, one for each possible ordering of the x paths. Thus, the number of such graphs is $\frac{\binom{j-1}{x-1}}{x!}j!$, as needed. □

In order to have a path $(b_1, b_2), (b_2, b_3), \ldots, (b_l, \text{center})$ in $G(t)$, a pair of elements from the columns corresponding to center must be placed into the column containing four b_1. The two displaced b_1 must then be placed into the same rows of the column containing four b_2. Continuing in this way, we are left with two b_l which replace the original pair of elements from center. Thus, to specify the volume of the columns corresponding to the paths in $G(t)$, it is necessary and sufficient to choose a pair from the columns corresponding to center for each path. Note that these pairs must be different as otherwise we would not end up with disjoint paths.

We now give the entire analysis for $D_{n,a,b}$. Given $A \subseteq [n]$ and $B \subseteq [n] \setminus A$, we can compute $D_{n,a,b} = \sum_{t \in D_{n,A,B}} \text{sgn}(t)|\mathcal{P}_{A,B}(t)|$ as follows. As before, we take $a = |A|$ and $b = |B|$.

1. For each $a \in A$, we choose which column contains six a. Similarly, for each $b \in B$, we choose which column contains four b. The number of choices for this is $\prod_{j=0}^{a+b-1}(n-j)$.

2. After choosing these columns, we choose a table $t \in T_{6,n-a-b}$ and a pairing $P \in \mathcal{P}(t)$ to fill in the remaining columns. This gives a factor of P_{n-a-b}.

3. We split into cases based on the number of vertices i in $G(t)$ which are contained in cycles. For each i, we choose which $\binom{b}{i}$ of the elements in B are contained in cycles. By Lemma 3.3.11, once these elements are chosen there are C_i possibilities for the columns containing these elements.

4. There are now $j = b - i$ elements of B which are contained in paths. We further split

into cases based on the number x of paths in $G(t)$. By Lemma 3.3.12, there are $\frac{\binom{j-1}{x-1}}{x!}j!$ possibilities for what these paths are in $G(t)$.

As discussed above, for each of the x paths we need to choose a different pair in P. The number of choices for these pairs is $\prod_{y=0}^{x-1}(3(n-a-b)-y)$. Summing all of these possibilities up gives a factor of

$$H_{n,j,a,b} = \sum_{x=1}^{j} \frac{\binom{j-1}{x-1}}{x!} j! \prod_{y=0}^{x-1} (3(n-a-b)-y).$$

Putting everything together, we have that

$$D_{n,a,b} = \left(\prod_{j=0}^{a+b-1}(n-j)\right)\left(\sum_{i=0}^{b}\binom{b}{i}C_i H_{n,b-i,a,b}\right)P_{n-a-b},$$

as needed. □

□

We now simplify the terms in Lemma 3.3.1.

Proposition 3.3.13.

$$H_{n,j,a,b} = \frac{(3(n-a-b)+j-1)!}{(3(n-a-b)-1)!}.$$

Proof. Originally,

$$H_{n,j,a,b} = \sum_{x=1}^{j} \frac{\binom{j-1}{x-1}}{x!} j! \prod_{y=0}^{x-1} (3(n-a-b)-y).$$

Denote $z = 3(n-a-b)$. For the inner product, we can write

$$\prod_{y=0}^{x-1}(3(n-a-b)-y) = \frac{z!}{(z-x)!},$$

so

$$H_{n,j,a,b} = \sum_{x=1}^{j} \frac{\binom{j-1}{x-1}}{x!} \frac{j!z!}{(z-x)!} = j! \sum_{x=1}^{j} \binom{j-1}{x-1} \binom{z}{x} = j! \binom{z+j-1}{j}.$$

The last equality is a special case of the **Chu-Vandermonde Identity**. □

Lemma 3.3.14. *Let S_n be the set of all permutations of order n and D_n the set of all* ***derangements*** *of the same order. That means, D_n is a subset of those permutations in S_n which have no fixed points. Denote $C(\pi)$ the number of cycles in a permutation π, then*

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\pi \in D_n} u^{C(\pi)} = \frac{e^{-ux}}{(1-x)^u}.$$

Proof. For a derivation, see the chapter on Bivariate generating functions in Flajolet and Sedgewick [2009]. □

Corollary 3.3.15.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} C_n = \frac{e^{-15x}}{(1-x)^{15}}.$$

Proof. An alternative way how to write C_n is via $C_n = \sum_{\pi \in D_n} 15^{C(\pi)}$. □

With these simplifications, we can derive an expression for the generating function

$$F_6(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_6(n).$$

By Lemma 3.3.1,

$$F_6(t) = \sum_{0 \le a \le j \le n} \frac{t^n}{(n!)^2} \binom{n}{j} \binom{j}{a} (m_6 - 15)^a (m_4 - 3)^{(j-a)} D_{n,a,j-a}.$$

Summing with respect to $b = j - a$ instead of a and observing that

$$\begin{aligned} D_{n,a,b} &= \left(\prod_{k=0}^{a+b-1} (n-k)\right) \left(\sum_{i=0}^{b} \binom{b}{i} C_i H_{n,b-i,a,b}\right) P_{n-a-b} \\ &= \frac{n!}{(n-j)!} \left(\sum_{i=0}^{b} \binom{b}{i} C_i H_{n,b-i,j-b,b}\right) P_{n-j} \\ &= n! \left(\sum_{i=0}^{b} \binom{b}{i} C_i H_{n,b-i,j-b,b}\right) \frac{(n-j+2)!(n-j+4)!}{48}, \end{aligned}$$

we have that

$$F_6(t) = \sum_{0 \leq i \leq b \leq j \leq n} \frac{t^n}{n!} \binom{n}{j} \binom{j}{b} \binom{b}{i} (m_6 - 15)^{(j-b)} (m_4 - 3)^b \frac{(n-j+2)!(n-j+4)!}{48} H_{n,b-i,j-b,b} C_i.$$

By Proposition 3.3.13, $H_{n,b-i,j-b,b} = (3n-3j+b-i-1)!/(3n-3j-1)!$. Using the reparametrization $b = i+s, j = b+r, n = j+q$, where s, r, q goes from 0 to ∞, we get

$$F_6(t) = \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^{i+s+r+q}}{q!r!s!i!} (m_6 - 15)^r (m_4 - 3)^{(i+s)} \frac{(q+2)!(q+4)!}{48} \frac{(3q+s-1)!}{(3q-1)!} C_i.$$

Grouping the terms to separate the dependence on r, s, and i, we have that $F_6(t)$ equals

$$\sum_{q=0}^{\infty} \frac{t^q}{q!} \frac{(q+2)!(q+4)!}{48} \left(\sum_{r=0}^{\infty} \frac{t^r (m_6-15)^r}{r!}\right) \left(\sum_{s=0}^{\infty} \frac{t^s}{s!} \frac{(3q+s-1)!}{(3q-1)!} (m_4-3)^s\right) \left(\sum_{i=0}^{\infty} \frac{t^i}{i!} (m_4-3)^i C_i\right).$$

Summing all the inner sums (the rightmost using Corollary 3.3.15),

$$F_6(t) = \sum_{q=0}^{\infty} \frac{t^q}{q!} \frac{(q+2)!(q+4)!}{48} e^{t(m_6-15)} \frac{1}{(1-t(m_4-3))^{3q}} \frac{e^{-15t(m_4-3)}}{(1-t(m_4-3))^{15}}.$$

3.3.1 Generalization for arbitrary third moment

Restating Proposition 2.2.5, we can write

$$f_6(n) = \sum_{t \in T_{6,n}} w(t)\,\mathrm{sign}(t),$$

where $T_{6,n}$ is the set of all permutation tables of length n with six rows (six-tables) whose columns fall in one of the following categories

- 6-columns: six copies of a single number (weight m_6)
- 4-columns: four copies of one number and two copies of a distinct number (weight m_4)
- 2-columns: three pairs of distinct numbers (weight 1)

The weight $w(t)$ of the table t is then simply a product of weights of its columns. To avoid ambiguity, we write $f_6^*(n)$ for $f_6(n)$ with m_3 being generally nonzero. We then define $T^*_{6,n}$ as the set of all six-tables having the following extra columns

- 3-columns: three copies of one number and three copies of a distinct number (weight m_3^2)

Similarly, it must hold that

$$f_6^*(n) = \sum_{t \in T^*_{6,n}} w(t)\,\mathrm{sgn}(t).$$

Proposition 3.3.16.

$$f_6^*(n) = \sum_{j=0}^{n} \binom{n}{j}^2 f_6(n-j)\, j!\, m_3^{2j} (-1)^j \sum_{\pi \in D_j} (-10)^{C(\pi)}.$$

Proof. The key is to group the summands according to the 3-columns in t. Those columns form a subtable s and the rest of the columns form another, complementary subtable t'. The signs of those tables are related as

$$\operatorname{sgn}(t) = \operatorname{sgn}(s)\operatorname{sgn}(t').$$

Denote $[n] = \{1, 2, 3, \ldots, n\}$. For a given $J \subset [n]$, we define $T_{6,J}$ a set of all six-tables of length $j = |J|$ composed with numbers in J. The set $T_{6,n}$ coincides with $T_{6,[n]}$. Denote $Q_{6,J}$ as the set of all six-tables composed only from 3-columns of numbers in J. We can write our sum, since the selection J does not depend on position in table t, as

$$f_6^*(n) = \sum_{J\subset[n]} \binom{n}{j} \sum_{t'\in T_{6,[n]/J}} w(t)\operatorname{sgn}(t) \sum_{s\in Q_{6,J}} w(s)\operatorname{sgn}(s).$$

No matter which numbers J are selected, as long as we select the same amount of them, the contribution is the same. Hence,

$$f_6^*(n) = \sum_{j=0}^{n} \binom{n}{j}^2 \sum_{t'\in T_{6,n-j}} w(t)\operatorname{sgn}(t) \sum_{s\in Q_{6,j}} w(s)\operatorname{sgn}(s),$$

where $Q_{6,j} = Q_{6,[j]}$. The first inner sum is simply $f_6(n-j)$. For the second inner sum, by symmetry, we can fix the first permutation in s to be identity. Upon noticing also that $w(s) = m_3^{2j}$, we get

$$\sum_{s\in Q_{6,j}} w(s)\operatorname{sgn}(s) = j!m_3^{2j} \sum_{\substack{s\in Q_{6,j}\\ s_1=\mathrm{id}}} \operatorname{sgn}(s).$$

We group the summands according to the following permutation structure: Let b be a number in the first row of a given column of table s. Since it is a 3-column, we denote the other number in the column as b'. We construct a permutation $\pi(s)$ to a given table s as composed from all those pairs $b \to b'$. Then

$$\operatorname{sgn}(s) = \operatorname{sign}(\pi(s)) = (-1)^{j-C(\pi(s))}.$$

Note that since b and b' are always different, the set off all $\pi(s)$ corresponds to the set D_j of all derangements. Since there are 10 possibilities how to arrange the leftover 5 numbers in the 3-columns corresponding to a given cycle of $\pi(s)$, we get

$$\sum_{s\in Q_{6,j}} w(s)\,\mathrm{sgn}(s) = j!m_3^{2j}(-1)^j \sum_{\pi\in D_j} (-1)^{C(\pi)} 10^{C(\pi)}.$$

and thus, all together

$$f_6^*(n) = \sum_{j=0}^{n} \binom{n}{j}^2 f_6(n-j)\, j! m_3^{2j}(-1)^j \sum_{\pi\in D_j} (-10)^{C(\pi)}.$$

□

Corollary 3.3.17.

$$F_6^*(t) = (1+m_3^2 t)^{10} e^{-10m_3^2 t}\, F_6(t).$$

Proof. In terms of generating functions,

$$\begin{aligned} F_6^*(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!^2} f_6^*(n) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{t^{n-j}}{(n-j)!^2} f_6(n-j) \frac{(-m_3^2 t)^j}{j!} \sum_{\pi\in D_j} (-10)^{C(\pi)} \\ &= F_6(t) \sum_{j=0}^{\infty} \frac{(-m_3^2 t)^j}{j!} \sum_{\pi\in D_j} (-10)^{C(\pi)} = F_6(t) \frac{e^{-10m_3^2 t}}{(1+m_3^2 t)^{-10}}. \end{aligned}$$

The final equality is a special case of Lemma 3.3.14. Theorem 2.1.6 follows. □

3.3.2 Asymptotics

The proof relies directly on the calculus developed by Borinsky [2018], enabling us to extract the asymptotic behaviour of coefficients from their factorially divergent generating function. We use the following result from Borinsky [2018]:

Definition 3.3.18. We say a formal power series $f(t) = \sum_{n\geq 0} f_n t^n$ is factorially divergent of type (α, β), if $f_n \sim \sum_{k=0}^{R} c_k \alpha^{n+\beta-k} \Gamma(n+\beta-k)$ as $n \to \infty$ for any fixed R integer. We also define an operator $\mathcal{A}_{\beta}^{\alpha}$ acting of $f(t)$ such that $(\mathcal{A}_{\beta}^{\alpha} f)(t) = \sum_{k \geq 0} c_k t^k$. If moreover $f(t)$ is analytic at 0, then $(\mathcal{A}_{\beta}^{\alpha} f)(t) = 0$.

Lemma 3.3.19. *Let $f(t)$ and $g(t)$ be two factorially divergent power series of type (α, β), then*

$$\begin{aligned} (\mathcal{A}_{\beta}^{\alpha}(fg))(t) &= (\mathcal{A}_{\beta}^{\alpha} f)(t) g(t) + f(t) (\mathcal{A}_{\beta}^{\alpha} g)(t), \\ (\mathcal{A}_{\beta}^{\alpha}(f \circ g))(t) &= f'(g(t)) (\mathcal{A}_{\beta}^{\alpha} g)(t) + \left(\frac{t}{g(t)} \right)^{\beta} e^{\frac{\frac{1}{t} - \frac{1}{g(t)}}{\alpha}} (\mathcal{A}_{\beta}^{\alpha} f)(g(t)), \end{aligned}$$

where the second equality holds when $g(t) = 1 + t + O(t^2)$.

Recall Theorem 2.1.6, which states

$$F_6(t) = \left(1 + m_3^2 t\right)^{10} \frac{e^{t\left(m_6 - 10 m_3^2 - 15 m_4 + 30\right)}}{48 \left(1 + 3t - m_4 t\right)^{15}} \sum_{i=0}^{\infty} \frac{(1+i)(2+i)(4+i)! t^i}{\left(1 + 3t - m_4 t\right)^{3i}}.$$

Hence, we can write $F_6(t) = h(t) f(g(t))$, where

$$\begin{aligned} f(t) &= \sum_{i=0}^{\infty} (1+i)(2+i)(4+i)! t^i, \\ g(t) &= \frac{t}{(1 + 3t - m_4 t)^3}, \qquad h(t) = \left(1 + m_3^2 t\right)^{10} \frac{e^{t\left(m_6 - 10 m_3^2 - 15 m_4 + 30\right)}}{48 \left(1 + 3t - m_4 t\right)^{15}} \end{aligned}$$

are factorially divergent of type $(1, 7)$ since

$$(1+i)(2+i)(4+i)! = \Gamma(i+7) - 8\Gamma(i+6) + 12\Gamma(i+5)$$

and $g(t)$ and $h(t)$ are analytic. Thus, by Lemma 3.3.19,

$$(\mathcal{A}_7^{\frac{1}{7}} F_6)(t) = h(t) \left(\frac{t}{g(t)}\right)^7 e^{\frac{1}{t}-\frac{1}{g(t)}} (\mathcal{A}_7^{\frac{1}{7}} f)(g(t)) = h(t) \left(\frac{t}{g(t)}\right)^7 e^{\frac{1}{t}-\frac{1}{g(t)}} (1 - 8g(t) + 12g^2(t)).$$

Apart from a factor $(n!)^2 e^{3(m_4-3)}/48$, this is our function $C(t)$ from the original statement of Theorem 2.1.9.

For $\Omega = \{-1, 1\}$, the asymptotic expression

$$f_6(n) \sim \frac{(n!)^3}{48e^6} \left(n^6 + 29n^5 + 335n^4 + \frac{5861n^3}{3} + \frac{17944n^2}{3} + \frac{44036n}{5} + \frac{167536}{45} - \frac{210176}{63n}\right)$$

gives an excellent approximation to $f_6(n)$ for $n \geq 10$. The following figure shows the ratio of this asymptotic expression to the actual value of $f_6(n)$ for n up to 20.

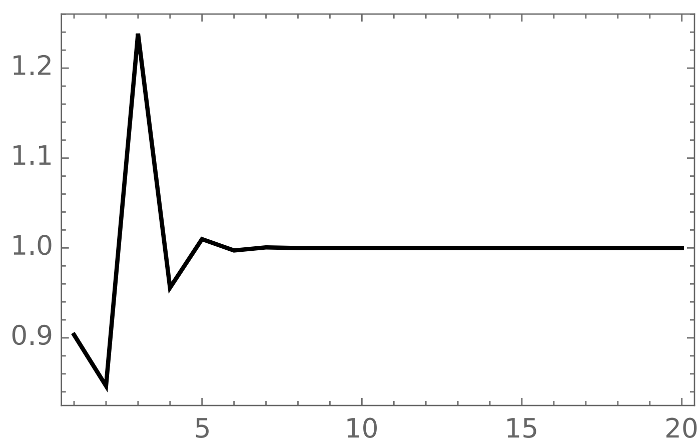

Figure 3.3: The ratio between the asymptotic expression and $f_6(n)$ for $\Omega = \{-1, 1\}$

3.3.3 Direct proof of Lemma 2.2.12

Lemma. *(Restatement of Lemma 2.2.12). For all $n \in \mathbb{N}$, $P_n = n(n+2)(n+4)P_{n-1}$ where $P_0 = 1$.*

Proof. We recursively compute $P_n = \sum_{t \in T_{k,n}} \mathrm{sgn}(t)|\mathcal{P}(t)|$ based on where the six n are located in t.

We can count the cases where all of the n are in a 6-column as follows. Given a table $t \in T_{k,n-1}$ and a pairing $P \in \mathcal{P}(t)$, we can obtain a table $t' \in T_{k,n}$ and a pairing $P' \in \mathcal{P}(t)$ by choosing a location for the 6-column, choosing a pairing for this column, and using t and P to fill in the remainder of t' and P'. There are n possible places for the 6-column, it has 15 possible pairings, and $\mathrm{sgn}(t') = \mathrm{sgn}(t)$, so this gives a contribution of $15nP_{n-1}$.

We can count the cases where four of the n are in a 4-column and two of the n appear in a different column as follows. Given a table $t \in T_{k,n-1}$ and a pairing $P \in \mathcal{P}(t)$, we can obtain a table $t' \in T_{k,n}$ and a pairing $P' \in \mathcal{P}(t)$ with the following steps:

1. Choose which column will be the 4-column containing four of the n. We initially put all six n in this column.

2. Fill in the remaining columns using t and P.

3. Choose one of the $3(n-1)$ pairs in P and swap two of the n with this pair.

4. Choose a pairing for the remaining four n.

There are n possible places for the 4-column containing four of the n, there are $3(n-1)$ pairs in P which can be swapped with two of the n, there are 3 different pairings for the remaining four n, and $\mathrm{sgn}(t') = \mathrm{sgn}(t)$, so this gives a contribution of $3*3*n(n-1)*P_{n-1} = 9n(n-1)P_{n-1}$.

The trickiest case to analyze is the case when the six n are split into three different columns. The idea for this case is that there is a correspondence between sets of 2 columns containing pairs of the elements a, b, c, d, e, f and sets of 3 columns containing pairs of the elements a, b, c, d, e, f where each column also contains a pair of n. This correspondence is highly non-trivial and relies on the signs of the permutations.

Definition 3.3.20. Let $S_1, S_2, S_3, S_4, S_5, S_6$ be six sets such that each set S_i contains two of the elements $\{a, b, c, d, e, f\}$ and each element in $\{a, b, c, d, e, f\}$ is contained in two of the sets $S_1, S_2, S_3, S_4, S_5, S_6$.

We define $T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ to be the set of 6×2 tables t such that the ith row contains the elements in S_i and each element appears an even number of times in each column. Similarly, we define $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$ to be the set of 6×3 tables t such that the ith row contains the elements in $S_i \cup \{n\}$ and each element appears an even number of times in each column.

For each $t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)$, we define $\text{sgn}(t)$ to be the product of the signs of the rows of t where row i of t has sign 1 if the elements of S_i appear in order and sign -1 if the elements of S_i appear out of order. Similarly, for each $t \in T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, we define $\text{sgn}(t)$ to be the product of the signs of the rows of t where row i of t has sign 1 if it takes an even number of swaps to transform it into $S_i \cup \{n\}$ and -1 if it takes an odd number of swaps to transform it into $S_i \cup \{n\}$.

Lemma 3.3.21. *For all possible $S_1, S_2, S_3, S_4, S_5, S_6$,*

$$\sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \text{sgn}(t) = 6 \sum_{t \in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \text{sgn}(t).$$

Corollary 3.3.22. *For all $n \in \mathbb{N}$,*

$$\sum_{t \in T_{6,n} : n \text{ appears in } 3 \text{ different columns}} \text{sgn}(t)|\mathcal{P}(t)| = n(n-1)(n-2)P_{n-1}.$$

Proof. Recall that

$$\sum_{t \in T_{6,n-1}} \text{sgn}(t)|\mathcal{P}(t)| = P_{n-1}.$$

We now apply Lemma 3.3.21 to the first two columns of the pairs (t, P) where $t \in T_{6,n-1}$ and $P \in \mathcal{P}(t)$. To do this, we use P to relabel the elements in the first two columns as a, b, c, d, e, f. One way to do this is as follows. We go through the rows one by one and assign the next unused label(s) to the element(s) which whose pair has not yet appeared. If there are two such elements, we assign the first unused label to the lower element and the next unused label to the higher element. If both elements are the same, we assign the first unused label to the column where the pair of this element appears first. If there is still a tie, we assign the same label to both elements and skip the next label. Lemma 3.3.21 still holds in this case as having $S_i = S_j = \{a, a\}$ instead of $S_i = S_j = \{a, b\}$ divides both sides by 2.

After doing this relabeling, for each $i \in [6]$, we take S_i to be the first two elements in row i. Applying Lemma 3.3.21, we obtain tables t' and pairings P' by taking P' to be the unique pairing for each column and inverting the labeling of the elements in the first two columns of t by $\{a, b, c, d, e, f\}$. This implies that whenever $n \geq 3$,

$$\sum_{t \in T_{6,n}: n \text{ appears in the first three columns}} \operatorname{sgn}(t)|\mathcal{P}(t)| = 6 \sum_{t \in T_{6,n-1}} \operatorname{sgn}(t)|\mathcal{P}(t)| = 6P_{n-1}.$$

There are $\binom{n}{3} = \frac{n(n-1)n-2)}{6}$ possibilities for which 3 columns contain n so we have that

$$\sum_{t \in T_{6,n}: n \text{ appears in 3 different columns}} \operatorname{sgn}(t)|\mathcal{P}(t)| = n(n-1)(n-2)P_{n-1},$$

as needed. □

Summing these three cases up, we have

$$\begin{aligned} P_n &= 15nP_{n-1} + 9n(n-1)P_{n-1} + n(n-1)(n-2)P_{n-1} \\ &= (n^3 + 6n^2 + 8n)P_{n-1} = n(n+2)(n+4)P_{n-1}. \end{aligned}$$

□

We now prove Lemma 3.3.21.

Proof of Lemma 3.3.21. Up to permutations of the rows and $\{a, b, c, d, e, f\}$, we have the following four cases for $S_1, S_2, S_3, S_4, S_5, S_6$:

1. $S_1 = S_2 = \{a, b\}$, $S_3 = S_4 = \{c, d\}$, and $S_5 = S_6 = \{e, f\}$.

2. $S_1 = S_2 = \{a, b\}$, $S_3 = \{c, d\}$, $S_4 = \{c, e\}$, $S_5 = \{d, f\}$, and $S_6 = \{e, f\}$.

3. $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, $S_3 = \{b, d\}$, $S_4 = \{d, e\}$, $S_5 = \{c, f\}$, and $S_6 = \{e, f\}$.

4. $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, $S_3 = \{b, c\}$, $S_4 = \{d, e\}$, $S_5 = \{d, f\}$, and $S_6 = \{e, f\}$.

We can see that these are the only possibilities as follows. If we construct a multi-graph where the vertices are $\{a, b, c, d, e, f\}$ and the edges are $\{S_1, S_2, S_3, S_4, S_5, S_6\}$ then in this multi-graph, every vertex will have degree 2.

1. If there is a cycle of length 2 then for the remaining 4 vertices, we will either have two more cycles of length 2 or a cycle of length 4. This gives cases 1 and 2.

2. If there is a cycle of length 3 then there must be another cycle of length 3 on the remaining vertices. This gives case 4.

3. If there are no cycles of length 2 or 3 then we must have a cycle of length 6. This gives case 3.

For the first three cases, $T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ is nonempty as shown by the examples below. For the fourth case, $T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ is empty.

$$\left\{\begin{matrix} a & b \\ a & b \\ c & d \\ c & d \\ e & f \\ e & f \end{matrix}\right\}, \left\{\begin{matrix} a & b \\ a & b \\ c & d \\ c & e \\ f & d \\ f & e \end{matrix}\right\}, \left\{\begin{matrix} a & b \\ a & c \\ d & b \\ d & e \\ f & c \\ f & e \end{matrix}\right\}$$

For all four cases, $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$ is nonempty as shown by the examples below.

$$\begin{Bmatrix} a & b & n \\ a & b & n \\ c & n & d \\ c & n & d \\ n & e & f \\ n & e & f \end{Bmatrix}, \begin{Bmatrix} a & b & n \\ a & b & n \\ c & n & d \\ c & n & e \\ n & f & d \\ n & f & e \end{Bmatrix}, \begin{Bmatrix} a & b & n \\ a & c & n \\ n & b & d \\ e & n & d \\ n & c & f \\ e & n & f \end{Bmatrix}, \begin{Bmatrix} a & b & n \\ a & n & c \\ n & b & c \\ d & e & n \\ d & n & f \\ n & e & f \end{Bmatrix}$$

We now show that for each of the four cases,

$$\sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \operatorname{sgn}(t) = 6 \sum_{t \in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \operatorname{sgn}(t).$$

1. For the first case, $|T_2(S_1, S_2, S_3, S_4, S_5, S_6)| = 8$ as we can choose the order of $\{a, b\}$ in row 1, the order of $\{c, d\}$ in row 3, and the order of $\{e, f\}$ in row 5. All $t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ have positive sign as rows 2, 4, and 6 must be the same as rows 1, 3, and 5. Thus, $\sum_{t \in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \operatorname{sgn}(t) = 8$.

   To analyze $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, observe that there are 6 choices for the positions of the n in rows 1, 3, and 5 and we can again choose the order of $\{a, b\}$ in row 1, the order of $\{c, d\}$ in row 3, and the order of $\{e, f\}$ in row 5. Thus, $|T_3(S_1, S_2, S_3, S_4, S_5, S_6)| = 48$. All $t \in T_3(S_1, S_2, S_3, S_4, S_5, S_6)$ have positive sign as rows 2, 4, and 6 must be the same as rows 1, 3, and 5 so we have that $\sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \operatorname{sgn}(t) = 48$.

2. For the second case, $|T_2(S_1, S_2, S_3, S_4, S_5, S_6)| = 4$ as we can choose the order of $\{a, b\}$ in row 1 and the order of $\{c, d\}$ in row 3 and this uniquely determines the rest of the table. It can be checked that all $t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ have positive sign so we have that $\sum_{t \in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \operatorname{sgn}(t) = 4$.

   To analyze $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, observe that there are 6 choices for the order of

$\{a, b, n\}$ in row 1. Once this order is chosen, there are two choices for the position of the n in row 3 and two choices for the order of $\{c, d\}$ in row 3. It can be checked that this uniquely determines the rest of the table and all $t \in T_3(S_1, S_2, S_3, S_4, S_5, S_6)$ have positive sign so we have that $\sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \text{sgn}(t) = 24$.

3. For the third case, $|T_2(S_1, S_2, S_3, S_4, S_5, S_6)| = 2$ as we can choose the order of $\{a, b\}$ in row 1 and this uniquely determines the rest of the table. Here both $t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ have negative sign so we have that $\sum_{t \in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \text{sgn}(t) = -2$.

   For $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, there are 6 choices for the order of $\{a, b, n\}$ in row 1. When row 1 is a, b, n, we have the following four tables:

$$\left\{\begin{matrix} a & b & n \\ a & c & n \\ n & b & d \\ e & n & d \\ n & c & f \\ e & n & f \end{matrix}\right\}, \left\{\begin{matrix} a & b & n \\ a & n & c \\ d & b & n \\ d & n & e \\ n & f & c \\ n & f & e \end{matrix}\right\}, \left\{\begin{matrix} a & b & n \\ a & n & c \\ n & b & d \\ e & n & d \\ n & f & c \\ e & f & n \end{matrix}\right\}, \left\{\begin{matrix} a & b & n \\ a & n & c \\ n & b & d \\ n & e & d \\ f & n & c \\ f & e & n \end{matrix}\right\}$$

   Of these tables, the first, second, and fourth table have negative sign while the third table has positive sign so the net contribution is -2. Multiplying this by 6, we have that

$$\sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \text{sgn}(t) = -12.$$

4. For the fourth case, $T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ is empty because each column can only contain one of $\{a, b, c\}$ and one of $\{b, c, d\}$.

   To analyze $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, observe that we can choose the order of $\{a, b, n\}$ in row 1 and the order of $\{d, e, n\}$ in row 4 and this uniquely determines the rest of the table. The sign of each table will be the product of the sign for row 1 and the sign for row 4, so we have the same number of tables with positive and negative sign and thus $\sum_{t\in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \text{sgn}(t) = \sum_{t\in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \text{sgn}(t) = 0$.

□

3.4 General Moments

In this section, we discuss the results of arbitrary k. To our knowledge, the formula for arbitrary k is only known to exist if each entry is an random Gaussian entry. The formula can be derived using a relatively widely recognized method involving the Wishart distribution, as mentioned by WILKS [1932]. An alternative demonstration can be found in the work of Nyquist et al. [1954]. In this section, we present the proof by Prékopa [1967]. Suppose that each entry of the random matrix is i.i.d sampled from $\mathcal{N}(0,1)$ and we show that $f^k(n) = \prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}$.

In order to prove the result, we first prove a lemma, which is based on the fact that the absolute value of determinant is the value of the volume of the 'parallelotope' formed by each vector and a property of the chi-squared distribution. In the following, we use χ^2_k to denote the chi-squared distribution with k degrees of freedom.

Lemma 3.4.1. *Let $v_1, ..., v_p$ be n-dimensional independent random vectors with independent components having standard normal distribution. The p-dimensional volume of the parallelotope Δ_n^p determined by these vectors is*

$$\Delta_n^p = \alpha_p \Delta_n^{p-1},$$

where α_k is sampled from a chi-distribution with $n-p+1$ degrees of freedom and Δ_n^{p-1} is distributed as the $p-1$-dimensional volume of $p-1$ independent random vectors having independent and standard normally distributed components.

Proof. First by 'base-times-heigh' formula of parallelotope, we use α_p to denote the orthogonoal part of v_p from the subspace spanned by $v_1, ..., v_{p-1}$, therefore we have $\Delta_n^p = \alpha_p \Delta_n^{p-1}$. Clearly α_p and Δ_n^{p-1} are two independent variables, and α_p is a χ-variable possessing $n - p + 1$ degrees of freedom, given that the subspace consisting of the initial $p - 1$ vectors can be established as the collection of points $(x_1, x_2, \ldots, x_n)$ where $x_p = x_{p+1} = \cdots = x_n = 0$, since Gaussian random vectors are rotationally invariant. □

Besides, we note the following result about the $k-$moment of chi square distribution from the textbook by Robert V. Hogg and Craig [2018], p.179:

Theorem 3.4.2. *The k-th moment of a χ^2-variable with $i-$degrees of freedom is equal to*

$$(i + 2k - 2)(i + 2k - 4)...(i + 2)i.$$

By combining 3.4.2 and 3.4.1, we have the formula for computing the $k-$th moment of $n \times n$ random matrix where each entry $m_{ij} \sim \mathcal{N}(0, 1)$.

Theorem 3.4.3. *(Restatement of Theorem 2.1.16) When k is even,*

$$f_k^{SN}(n) = \prod_{j=0}^{\frac{k}{2}-1} \frac{(n + 2j)!}{(2j)!}.$$

CHAPTER 4

MOMENTS OF DETERMINANTS OF RANDOM SYMMETRIC MATRICES

4.1 Preliminaries

Now we introduce some tools for dealing with the determinants of random symmetric matrices. Instead of using tables, here we use graph to capture the symmetry between vertices.

Definition 4.1.1. Given natural numbers k and n where k is even, we define a even graph G to be a directed graph $G = (V, E, C)$ where $V = [n]$ and $\deg^+(v) = \deg^-(v) = k$ for all $v \in V$. $C : E \to [k]$ where the number of edges colored with $i \in [k]$ is n, and each connected components in the edge-induced subgraph of the same color class is a directed cycle. Furthermore, we require that each edges must appear in pairs regardless of direction and color. We define $\mathcal{G}_{k,n}$ to be the set of all even graphs with n vertices and kn edges.

Definition 4.1.2. Given an even graph G of n vertices and kn edges, we define its sign $\text{sgn}(G)$ to be the product of the signs of its cycles, where each color edge-induced subgraph is a permutation of $[n]$.

Definition 4.1.3. Given an ordered pair of vertices i, j of G, we define its weight $w(i, j)$ to be

$$w(c) = m_{\#\text{edges between } i,j}.$$

Definition 4.1.4. Given an even graph G of n vertices and kn edges, we define its weight $w(G)$ to be the product of the weights of its pairs of vertices.

We can use the following proposition to reduce the problem of finding the sixth moment of a random determinant to a combinatorial problem.

Proposition 4.1.5. *For all even $k \in \mathbb{N}$, $f_k^{sym}(n) = \sum_{G \in \mathcal{G}_{k,n}} \mathrm{sgn}(G) w(G)$ and $p_k^{sym}(n) = \sum_{G \in \mathcal{G}_{k,n}} \mathrm{sgn}(G) w(G)$.*

Proof. The proof is same to that of asymmetric case but note that here $x_{ij} = x_{ji}$. □

Remark 4.1.6. $f_2^{sym} = p_2^{sym}$.

Example 4.1.7. The following graph corresponds to $\{2, 1, 5, 3, 4\}, \{2, 1, 4, 5, 3\}$.

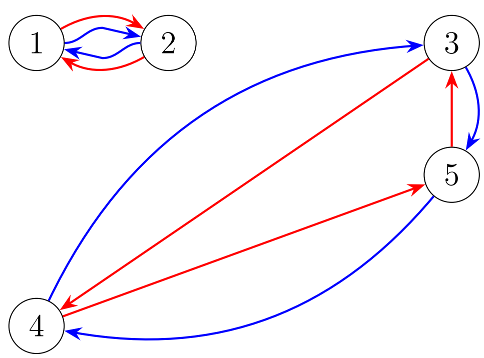

Figure 4.1: The even graph corresponds to $\{2, 1, 5, 3, 4\}, \{2, 1, 4, 5, 3\}$

4.2 Second Moment

The following result on the second moment of determinants for random symmetric matrices is obtained through a similar proof of our proof of 2.1.2. We begin by reducing the problem to counting a kind of combinatorial structures with a certain set of properties, and then use generating function to find the explicit formula.

Theorem 4.2.1. *For any distribution Ω such that $m_1 = 0$ and $m_2 = 1$,*

$$f_2^{sym}(n) = n! \sum_{n=0}^{\infty} \sum_{0 \le q \le \lfloor \frac{n}{2} \rfloor} \prod_{i=1}^{2q} \frac{2i-1}{2i} \sum_{0 \le p \le n-2q} (n-2q-p+1) \sum_{k=\lceil \frac{p}{2} \rceil}^{p} \frac{(-1)^k}{(p-k)!(2k-p)!} \left(\frac{3-m_4}{2} \right)^{p-k}.$$

Proof. By 4.1.5, we know the value of $f_2^{sym}(n)$ is the number of elements in $\mathcal{G}_{2,n}$. To count number of directed graphs which satisfy the constraints, we classify them based on how many vertices are in the connected component that includes vertex 1.

For a single connected component with l vertices, we first consider the number of cycles that can be made with these vertices then we copy the edges into pairs, orient and color them. The number of undirected cycles in l vertices is $\frac{l!}{2l} = \frac{(l-1)!}{2}$. For orientation and coloring given an undirected cycle, we first discuss the case that $l \geq 4$ and l is even. We first color them accordingly, we can obtain two kinds of colorings, where in one coloring each of the colored edges forms a cycle and in the other each of them forms a perfect matching. For the first, we can orient each of them clockwise or anti-clockwise, therefore we have 4 different orientations. For the second case, since we have two colorings and two different matching, we have two ways to assign the colors to each of the matching and once we have them in the matching form, the orientation is fixed. So for $l \geq 4$ and l is even, we have $3(l-1)!$ different connected components with l vertices. We can use a similar proof to get $2(l-1)!$ in the case that l is odd. Note that when $l = 2$, we have four edges corresponds to the same variable, so in this case, the weight is m_4, which is the fourth moment of the variable. Let $K(l)$ be the function to denote the number of different connected components with l vertices, so we have

$$K(n) = \begin{cases} 1 & l \leq 1 \\ m_4 & l = 2 \\ 2(l-1)! & l \geq 3, l \text{ odd} \\ 3(l-1)! & l \geq 4, l \text{ even} \end{cases}$$

Example 4.2.2. Let G be an undirected cycle graph with four vertices. We show all the 6 different colorings and orientations given G.

For the total number of such graphs, we can iterate on how many vertices are in the connected component that include vertex 1. If there l vertices in this component, the number of different such directed and colored graphs of the remaining $n-l$ vertices is just $f_2^{sym}(n-l)$.

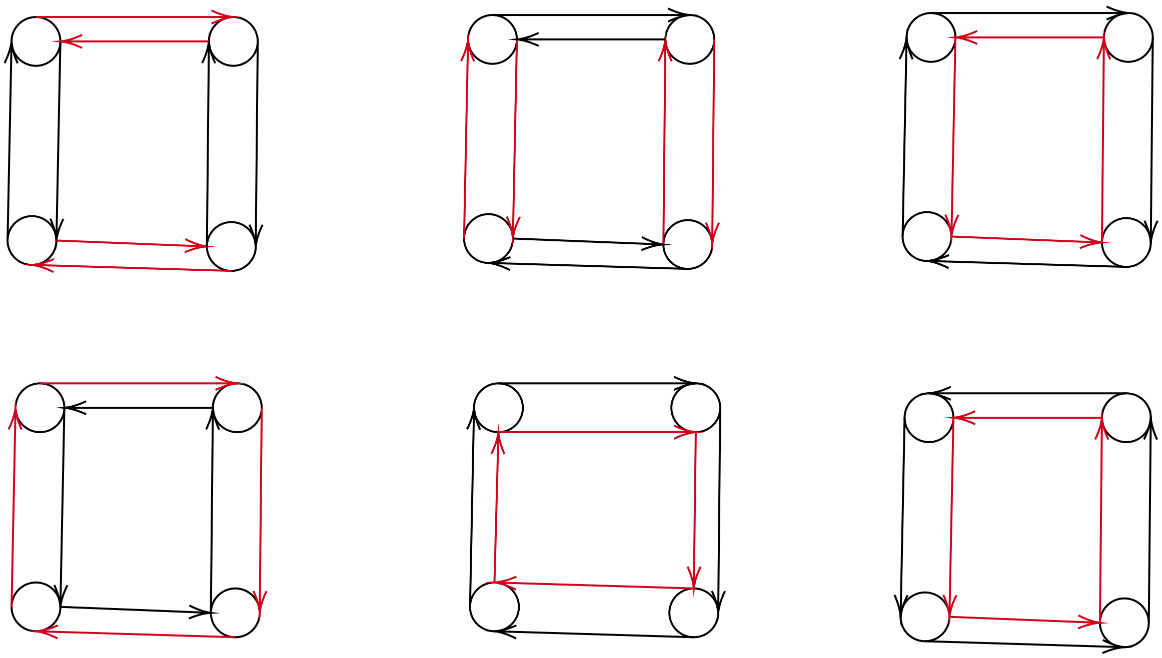

Figure 4.2: All 6 different colorings and orientations of a given 4−cycle.

Therefore, by summing over all possible value of l, we have w

$$f_2^{sym}(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} K(i) f_2^{sym}(n-i),$$

where $f_2^{sym}(1) = 1, f_2^{sym}(0) = 1$.

By rearranging the terms, we get

$$f_2^{sym}(n) = q(n) * (n-1)!,$$

where

$$p(n) = \begin{cases} 0 & n = 0 \\ 1 & 0 < n \leq 1 \\ m_4 & n = 2 \\ 2 & n \geq 3, n \text{ odd} \\ 3 & n \geq 4, n \text{ even} \end{cases}$$

$$q(n) = p(n) + \sum_{i=1}^{n-1} \frac{p(i)q(n-i)}{n-i}.$$

The next step to find the explicit formula of $f_2^{sym}(n)$ by using ordinary generating functions.

First we denote $r(n) := \frac{q(n)}{n}$. Therefore, $f_2^{sym}(n) = n!r(n)$, then by arranging the terms in $r(n)$, we obtain the following recurrence with base case $r(0) = 1$:

$$nr(n) = \sum_{i=0}^{n} p(i)r(n-i).$$

By multiplying both sides with x^n and taking sum over all n, we have

$$\sum_{n\geq 0}^{\infty} nr(n)x^n = \sum_{n\geq 0}^{\infty}\sum_{i=0}^{n} p(i)r(n-i)x^n.$$

Denote the ordinary generating function of $r(n)$ as $R(x) = \sum_{n\geq 0}^{\infty} r(n)x^n$. We have the following differential equation of $R(x)$:

$$xR'(x) = R(x)P(x).$$

Using the fact that $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$, the formula of $P(x)$ is easily obtained as:

$$P(x) = x + m_4 x^2 + \frac{2x^3}{1-x^2} + \frac{3x^4}{1-x^2}.$$

By plugging the formula of $P(x)$ and solving the differential equation, we have the generating function of $R(x)$:

$$R(x) = \frac{e^{\frac{1}{2}x(-2+(-3+m4)x)}}{(1-x)^{\frac{5}{2}}\sqrt{1+x}}$$

Then our next step is to find the formula for $r(n)$ explicitly. First by rearranging the terms in $R(x)$, we get

$$R(x) = \frac{e^{-(\frac{3-m_4}{2})x^2 - x}}{(1-x)^2\sqrt{1-x^2}}.$$

Now let's consider the series expansion of each term:

- $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$.
- $\frac{1}{\sqrt{1-x^2}}$. First we notice that $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{1*3x^2}{2*4} + \frac{1*3*5x^3}{2*4*6} + \frac{1*3*5*7x^3}{2*4*6*8}...$, which can be written as
$$\sum_{n=0}^{\infty}\left(\prod_{i=1}^{n}\frac{2i-1}{2i}\right)x^{2n}.$$
-

$$\begin{aligned}
e^{-(\frac{3-m_4}{2})x^2 - x} &= \sum_{k=0}^{\infty}\frac{(-(\frac{3-m_4}{2})x^2 - x)^k}{k!} \\
&= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\binom{k}{n}(\frac{3-m_4}{2}x)^n\right)\frac{(-1)^k}{k!}x^k \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n}\frac{(-1)^k}{(n-k)!(2k-n)!}(\frac{3-m_4}{2})^{n-k}\right)x^n
\end{aligned}$$

In order to merge them together, we use the generating function convolution on these three terms and obtain the following expansion:

$$\frac{e^{-\left(\frac{3-m_4}{2}\right)x^2-x}}{(1-x)^2}=\sum_{n=0}^{\infty}\left(\sum_{0\le p\le n}\sum_{k=\lceil\frac{p}{2}\rceil}^{p}\frac{(-1)^k}{(p-k)!\,(2k-p)!}\left(\frac{3-m_4}{2}\right)^{p-k}(n+1-p)\right)x^n$$

$$R\,(x)=\sum_{n=0}^{\infty}\left(\sum_{0\le q\le\lfloor\frac{n}{2}\rfloor}\prod_{i=1}^{2q}\frac{2i-1}{2i}\sum_{0\le p\le n-2q}(n-2q-p+1)\sum_{k=\lceil\frac{p}{2}\rceil}^{p}\frac{(-1)^k}{(p-k)!\,(2k-p)!}\left(\frac{3-m_4}{2}\right)^{p-k}\right)x^n.$$

Therefore, we have

$$f_2^{sym}(n)=n!\sum_{n=0}^{\infty}\sum_{0\le q\le\lfloor\frac{n}{2}\rfloor}\prod_{i=1}^{2q}\frac{2i-1}{2i}\sum_{0\le p\le n-2q}(n-2q-p+1)\sum_{k=\lceil\frac{p}{2}\rceil}^{p}\frac{(-1)^k}{(p-k)!(2k-p)!}\left(\frac{3-m_4}{2}\right)^{p-k}.$$

□

As noted in Section 2.1.2, Zhurbenko [1968] obtained the following asymptotics.

Theorem 4.2.3. *For any distribution Ω such that $m_1=0$ and $m_2=1$,*

$$f_2^{sym}(n)=C_n n^{\frac{3}{2}}n!$$

where $\lim_{n\to\infty}C_n=\frac{4\sqrt{2\pi}e^{-2}}{3}$.

CHAPTER 5

OPEN PROBLEMS

In this section we outline several interesting directions for future work on the related topics.

5.1 Higher Moments of Random Determinants

The first general extension in this direction is to find formulas for higher moments, such as the eighth moment. For the eighth moment, our techniques might not be directly applied due to the complicated structures induced by the tables T_8. Therefore, more insights are needed to improve the current result. Moreover, there is a chance that the direct formula of the eighth moment might not exist, so finding the asymptotic behavior would be a good research problem to focus on.

Another direction we would like to explore is finding the sixth moment of random determinants where each entry is i.i.d sampled from an arbitrary distribution instead of a symmetric distribution.

Moreover, it would be nice if our result for the sixth moment is generalized for $p \times n$ matrices, where we consider $E\left[\det(MM^T)^{\frac{k}{2}}\right]$ rather than $E\left[\det(M)^k\right]$.

The most challenging and intriguing work in this direction is to find a general formula $F(n, k)$ that computes the $k-$th moment of determinants of random $n \times n$ matrices. As noted in Section 3.4, when each entry is sampled from the Gaussian distribution, we have the formula $F_G(n, k) = \prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}$. We wonder if such a formula exists for arbitrary distributions. A potential first step is to find such a formula for random matrices where each entry is i.i.d sampled from the uniform distribution over ± 1. Note that in the latter case, $F(3, 2k) = 6 * 16^{k-1}$.

We believe that finding finding the exact formulas for the problems mentioned above might be challenging. Therefore, it would be nice to find the approximate formulas.

5.2 Determinants of Symmetric Random Matrices

A natural problem to consider is finding the formula for the fourth moment of symmetric random matrices. However, with our current techniques, there would be some difficulties. While counting the number of 2-regular graphs is straightforward, counting the number of 4-regular graphs with additional properties should be challenging, to our understanding. Therefore, to further improve the result, either a new structural insight is needed, or we can find the asymptotic behavior by approximating the number of 4-regular graphs.

Another open problem is to study higher moments of determinants for symmetric random matrices with entries from other distributions, such as Gaussian or uniform distributions. This would provide a more complete understanding of the relationship between the moments of determinants and the underlying distribution of the matrix entries.

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